A kinetic equation for unstable plasmas in a finite space-time domain

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(Received 13 June 2008; accepted 19 August 2008; published online 19 September 2008)

A generalization of the Lenard–Balescu equation is derived that includes wave-particle scattering by collective instabilities in finite space-time domain plasmas. It is shown that wave-particle interactions can dominate conventional particle-particle Coulomb scattering when finite times are considered or convective instabilities are present in a medium of finite spatial extent before the instabilities grow to nonlinear or turbulent amplitudes. The modified Lenard–Balescu operator retains important physical properties including particle, momentum, and energy conservation laws and the Boltzmann $\mathcal{H}$-theorem. To demonstrate its utility, the theory is applied to the simple example of convectively growing ion-acoustic instabilities in a finite spatial domain. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2979689]

I. INTRODUCTION

A collision operator for the linearized plasma kinetic equation that describes nonequilibrium behavior in the presence of a class of collective instabilities is developed. Most previous theories either describe the stable plasma limit of these linear equations, e.g., the Lenard–Balescu equation,[1,2] or resort to solving the nonlinear problem, cf. Ref. 3. However, if convective instabilities are present in a finite plasma, or simply for a finite time in an absolutely unstable plasma, instabilities may be virulent enough to enhance scattering but not reach a nonlinear level where turbulence theories are required. In this "quasiclassical" regime,[4] a linear analysis is warranted.

Plasma fluctuations and convective modes have been studied previously by Kent and Taylor[5] for an inhomogeneous plasma with a magnetic field using the "dressed particle method." Enhanced electron scattering due to convective modes excited by the loss-cone instability in magnetic mirror machines was explored by Baldwin and Callen.[6] In this work we use the "dressed particle method" to derive a collision operator for unmagnetized plasmas and emphasize generality before applying the theory to a specific problem where the ion-acoustic instability is present. We show that the presence of instabilities gives rise to a long range correlation between particles that extends well beyond the conventional Debye sphere to which Coulomb interactions are confined in stable plasmas. The analysis is shown to be valid over a finite time or space domain depending on what type of instabilities, i.e., absolute or convective, are present.

Amplification of the scattering rate, similar to that derived in Ref. 6, is found for virulent convective instabilities. This work emphasizes similarities between the resultant collision operator and the conventional Lenard–Balescu collision operator. Recognition of this similarity aids in deriving properties a physically meaningful collision operator should possess such as conservation laws and the Boltzmann $\mathcal{H}$-theorem. By doing so, we generalize the Lenard–Balescu equation, which is typically referred to as a highly accurate correction of the Landau collision operator,[7,8] to describe scattering by a fundamentally different physical mechanism that can be applied to an entirely new class of problems. Problems of interest involving convective instabilities include collisions in the presence of the ion-acoustic instability,[9] non-Maxwellian plasmas,[10] and anomalous resistivity.[11,12] Previous studies have relied largely on numerical methods or quasilinear theory. In this work, the role of a finite-domain (time or space) is emphasized.

This paper is organized as follows: Sec. II contains a derivation of a collision operator for stable, as well as unstable, plasmas. Amplification of scattering by unstable modes is interpreted for a finite time or space domain depending on the type of instability in Sec. II B. The validity of the linear model is discussed in Sec. III. Section IV lists important physical properties of the collision operator with proofs where relevant. As an example, the ion-acoustic instability is considered in Sec. V to illustrate enhanced scattering by convective instabilities. Section VI summarizes the paper's conclusions.

II. KINETIC EQUATION

The plasma kinetic equation can be derived from the Klimontovich equation, $dF/dt=0$, by an appropriate average of the "exact" distribution function

$$F \equiv \sum_{i}^{N} \delta[\mathbf{x}-\mathbf{x}_i(t)]\delta[\mathbf{v}-\mathbf{v}_i(t)]. \tag{1}$$

Here $\mathbf{x}$ and $\mathbf{v}$ are the phase space coordinates while $\mathbf{x}_i$ and $\mathbf{v}_i$ represent the position of particle i in phase space. The quantity d/dt is the convective derivative in the six-dimensional phase space $(\mathbf{x},\mathbf{v})$. The plasma kinetic equation then follows from separating the smoothed and discrete particle components of F, $F=f+\delta f$, and the electromagnetic fields, where $f\equiv\langle F\rangle$ and $\langle\delta f\rangle=0$. Here, the bracket denotes an ensemble average. The desired plasma kinetic equation is obtained from an ensemble average of the Klimontovich equation us-

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ing a linear closure scheme to determine the particle-discreteness distribution δf.[13,14]

In this paper we assume there is no "ensemble averaged," i.e., equilibrium, electric or magnetic fields, $\langle \mathbf{E} \rangle = 0$ and $\langle \mathbf{B} \rangle = 0$, and only electrostatic perturbations are present so $\delta \mathbf{B} = 0$. With these assumptions the plasma kinetic equation is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} = C(f), \tag{2}$$

where the collision operator is[13,14]

$$C(f) \equiv -\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{J}_v \quad \text{and } \mathbf{J}_v \equiv \frac{q}{m} \langle \delta \mathbf{E} \delta f \rangle \tag{3}$$

is the collisional current induced by particle-discreteness effects.

An equation for δf is obtained by subtracting Eq. (2) from the Klimontovich equation,

$$\frac{d\delta f}{dt} = -\frac{q}{m}\left(\delta \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} + \delta \mathbf{E} \cdot \frac{\partial \delta f}{\partial \mathbf{v}} - \left\langle \delta \mathbf{E} \cdot \frac{\partial \delta f}{\partial \mathbf{v}} \right\rangle \right). \tag{4}$$

A linearized form of Eq. (4), which neglects the last two terms, is used to derive the conventional Lenard–Balescu collision operator for a stable plasma. For a stable plasma, the last two terms of Eq. (4) can be rigorously shown to be smaller than the linear terms by $\mathcal{O}(1/n\lambda_{\mathrm{De}}^3) \ll 1$.[15]

The linearized form of Eq. (4) can also be applied to unstable plasmas in a finite space-time domain as long as the fluctuation level in the unstable region of the plasma remains small enough. Formally the requirement is

$$\left| \delta \mathbf{E} \cdot \frac{\partial \delta f}{\partial \mathbf{v}} - \left\langle \delta \mathbf{E} \cdot \frac{\partial \delta f}{\partial \mathbf{v}} \right\rangle \right| \ll \left| \delta \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} \right|, \tag{5}$$

to justify the linear theory. In Sec. III we show that for $\omega \ll k v_{Te}$ this is equivalent to $q\delta\phi/T_e \lesssim 1$. Absolute instabilities must be confined to a finite time domain and convective instabilities to a finite space domain. If instabilities are allowed to grow over a long enough domain to violate Eq. (5), then nonlinear or turbulence methods must be used.[3]

In the following, the linear form of Eq. (4),

$$\frac{d\delta f}{dt} = -\frac{q}{m} \delta \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} \tag{6}$$

is used for each species s along with Gauss' law,

$$\frac{\partial}{\partial \mathbf{x}} \cdot \delta \mathbf{E} = 4\pi \sum_s q_s \int d^3v \, \delta f, \tag{7}$$

to derive a collision operator, $C(f)$, that is valid for unmagnetized plasmas that are either stable or unstable in a finite space-time domain.

A. Collision operator

To solve for the collision operator, we employ the method of characteristics along with a combined Fourier transform in space and Laplace transform in time. In the absence of equilibrium electric and magnetic fields, the characteristics are the free particle trajectories $\mathbf{v}' = \mathbf{v}$ and $\mathbf{x}' = \mathbf{x} + \mathbf{v}(t' - t)$ with initial conditions $\mathbf{x}'(t' = t) = \mathbf{x}$ and $\mathbf{v}'(t' = t) = \mathbf{v}$. Integrating Eq. (6) along the characteristics gives

$$\delta f(\mathbf{x}, \mathbf{v}, t) = \delta f(t' = 0) - \frac{q}{m} \int_0^t dt' \, \delta \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}'}, \tag{8}$$

where $\delta f(t' = 0)$ is the initial condition of the "exact" distribution $\delta f = F - f$. Fourier and Laplace transforming Eq. (8) and inserting the characteristic equations yields an expression for the transformed distribution perturbation

$$\delta \hat{f}(\mathbf{k}, \mathbf{v}, \omega) = \frac{i \delta \tilde{f}(\mathbf{k}, \mathbf{v}, t' = 0)}{\omega - \mathbf{k} \cdot \mathbf{v}} - \frac{q}{m} \mathbf{k} \cdot \frac{\partial f}{\partial \mathbf{v}} \frac{\delta \hat{\phi}}{\omega - \mathbf{k} \cdot \mathbf{v}}, \tag{9}$$

where the "hat" denotes Fourier and Laplace transformed variables and the "tilde" denotes only Fourier transformed variables. We have also written $\delta \mathbf{E}$ in terms of the electric potential, $\delta \mathbf{E}(\mathbf{x}, t) = -\partial \delta \phi(\mathbf{x}, t) / \partial \mathbf{x}$.

Substituting Eq. (9) into the Fourier–Laplace transform of Gauss's law, Eq. (7), leads to

$$\delta \hat{\phi} = \sum_s \frac{4\pi q_s}{k^2 \hat{\varepsilon}} \int d^3v \frac{i \delta \tilde{f}(t' = 0)}{\omega - \mathbf{k} \cdot \mathbf{v}}, \tag{10}$$

where

$$\hat{\varepsilon} = 1 + \sum_s \frac{4\pi q_s^2}{k^2 m_s} \int d^3v \frac{\mathbf{k} \cdot \partial f_s / \partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} \tag{11}$$

is the familiar dielectric function for an equilibrium-field-free plasma. Equation (10) can be simplified by substituting in

$$\delta \tilde{f}(t' = 0) = \sum_{i=1}^N e^{-i\mathbf{k} \cdot \mathbf{x}_{io}} \delta(\mathbf{v} - \mathbf{v}_{io}) - (2\pi)^3 \delta(\mathbf{k}) f, \tag{12}$$

where $\mathbf{v}_{io} \equiv \mathbf{v}(t = 0)$, to give

$$\delta \hat{\phi} = \sum_{s,i=1}^N \frac{4\pi q_s}{k^2 \hat{\varepsilon}} \frac{i e^{-i\mathbf{k} \cdot \mathbf{x}_{io}}}{\omega - \mathbf{k} \cdot \mathbf{v}_{io}}. \tag{13}$$

Here we have used for the initial conditions that F satisfies $F(t = 0) = \Sigma_i \delta(\mathbf{x} - \mathbf{x}_{io}) \delta(\mathbf{v} - \mathbf{v}_{io})$ and that the Fourier terms of f are given by $(2\pi)^3 \delta(\mathbf{k}) f$; f is essentially uniform in space relative to spatial scales of δf. The term involving f produces no contribution to $\delta \phi$ due to quasineutrality,

$$\sum_s q_s \int d^3v \frac{\delta(\mathbf{k}) f}{\omega - \mathbf{k} \cdot \mathbf{v}} = \frac{\delta(\mathbf{k})}{\omega} \sum_s n_s q_s = 0. \tag{14}$$

Using Eqs. (12) and (13), we find an expression for $\delta \hat{f}(\mathbf{k}, \mathbf{v}, \omega)$ from Eq. (9),

$$\delta \hat{f} = \sum_{s,i=1}^N \left[\frac{i e^{-i\mathbf{k} \cdot \mathbf{x}_{io}} \delta(\mathbf{v} - \mathbf{v}_{io})}{\omega - \mathbf{k} \cdot \mathbf{v}} - \frac{i (2\pi)^3 \delta(\mathbf{k}) f}{\omega - \mathbf{k} \cdot \mathbf{v}} - \frac{4\pi q q_s}{m k^2 \hat{\varepsilon}} \frac{i \mathbf{k} \cdot \partial f / \partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} \frac{e^{-i\mathbf{k} \cdot \mathbf{x}_{io}}}{\omega - \mathbf{k} \cdot \mathbf{v}_{io}} \right], \tag{15}$$

which along with Eq. (13) determines the transform of the collision operator.

Since the transform of the collisional current, $\hat{\mathbf{J}}_v$, is the ensemble average of the convolution of electric field and distribution perturbations, it is convenient to define different transform variables for $\delta\hat{\mathbf{E}}$ and $\delta\hat{f}$. Keeping the notation of Eq. (15) the same and changing that of Eq. (13), we write

$$\delta\hat{\mathbf{E}}(\mathbf{k}',\omega') = \sum_{s,l=1}^{N} \frac{4\pi q_s}{k'^2\hat{\varepsilon}(\mathbf{k}',\omega')} \frac{\mathbf{k}' e^{-i\mathbf{k}'\cdot\mathbf{x}_{lo}}}{\omega' - \mathbf{k}'\cdot\mathbf{v}_{lo}}. \tag{16}$$

Then the transformed collisional current is defined by

$$\hat{\mathbf{J}}_v(\mathbf{k},\mathbf{k}',\mathbf{v},\omega,\omega') = \frac{q}{m}\langle \delta\hat{\mathbf{E}}(\omega',\mathbf{k}')\delta\hat{f}(\omega,\mathbf{k},\mathbf{v})\rangle, \tag{17}$$

where the ensemble average is[16]

$$\langle \ldots \rangle \equiv \prod_{l=1}^{N} \int d^3x_{lo} d^3v_{lo} \frac{f(\mathbf{v}_{lo})}{nV}(\ldots), \tag{18}$$

in which n denotes density and V denotes volume.

Taking the ensemble average of the product of Eqs. (15) and (16) gives an array of terms,

$$\hat{\mathbf{J}}_v = \frac{q}{m}\prod_{l=1}^{N} \int d\Gamma_l \sum_{i=1}^{N} \delta\hat{\mathbf{E}}_l\left[g_i - \frac{i(2\pi)^3\delta(\mathbf{k})f}{\omega - \mathbf{k}\cdot\mathbf{v}}\right], \tag{19}$$

where

$$d\Gamma_l \equiv \frac{d^3x_{lo}d^3v_{lo}}{nV} f(\mathbf{v}_{lo}) \tag{20}$$

and g_i is the first and last terms of Eq. (15). For unlike particle terms, $i \neq l$, the $\mathbf{x}_{lo}$ integral yields $(2\pi)^3\delta(\mathbf{k}')$. Since the rest of these terms tend to zero in the limit that $\mathbf{k}' \to 0$, the "unlike" particle terms vanish upon inverse Fourier transforming.[17] By the same argument the last term vanishes as well. We are then left with only "like" particle correlations after the ensemble average. For these $i=l$ terms the $\mathbf{x}_{lo}$ integral yields

$$\int d^3x_{io} \exp[-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}_{io}] = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}'). \tag{21}$$

Since only "like" particles remain, $\mathbf{v}_{io}$ is a dummy variable of integration in the resulting equation and the sum over all particles becomes simply the total number of particles in the volume, $\Sigma_{i=1}^{N}/V = N/V = n$. Labeling $\mathbf{v}_{io} = \mathbf{v}'$, the transformed collisional current is found to be

$$\hat{\mathbf{J}}_v = \frac{4\pi q^2}{mk^2}\int d^3v' f(\mathbf{v}') \frac{i\mathbf{k}'(2\pi)^3\delta(\mathbf{k}+\mathbf{k}')}{\hat{\varepsilon}(\mathbf{k}',\omega')(\omega' - \mathbf{k}'\cdot\mathbf{v}')} \times\left[\frac{\delta(\mathbf{v}-\mathbf{v}')}{\omega - \mathbf{k}\cdot\mathbf{v}} - \frac{4\pi}{k^2 m}\mathbf{k} \cdot\frac{\partial f(\mathbf{v})}{\partial \mathbf{v}} \frac{q_s^2}{(\omega - \mathbf{k}\cdot\mathbf{v})(\omega - \mathbf{k}\cdot\mathbf{v}')\hat{\varepsilon}(\mathbf{k},\omega)}\right]. \tag{22}$$

Symmetry between the two terms in this expression becomes explicit by evaluating the trivial $\mathbf{v}'$ integral in the first term, then multiplying this term by $\hat{\varepsilon}/\hat{\varepsilon}$ where the numerator is written in terms of Eq. (11) with $\mathbf{v} \leftrightarrow \mathbf{v}'$,

$$\hat{\mathbf{J}}_v = \frac{(4\pi)^2 q^2 q_s^2}{mk^4}\int d^3v' \frac{i\mathbf{k}'(2\pi)^3\delta(\mathbf{k}+\mathbf{k}')}{\hat{\varepsilon}(\mathbf{k}',\omega')\hat{\varepsilon}(\mathbf{k},\omega)(\omega - \mathbf{k}\cdot\mathbf{v})(\omega - \mathbf{k}\cdot\mathbf{v}')} \times\left[\frac{f(\mathbf{v})\mathbf{k}\cdot\partial f(\mathbf{v}')/\partial\mathbf{v}'}{m_s(\omega' - \mathbf{k}'\cdot\mathbf{v})} - \frac{f(\mathbf{v}')\mathbf{k}\cdot\partial f(\mathbf{v})/\partial\mathbf{v}}{m(\omega' - \mathbf{k}'\cdot\mathbf{v}')}\right] + \frac{4\pi q^2}{mk^2} f(\mathbf{v}) \frac{i\mathbf{k}'\delta(\mathbf{k}+\mathbf{k}')(2\pi)^3}{\hat{\varepsilon}(\mathbf{k}',\omega')(\omega' - \mathbf{k}'\cdot\mathbf{v})(\omega - \mathbf{k}\cdot\mathbf{v})\hat{\varepsilon}(\mathbf{k},\omega)}. \tag{23}$$

The last term in Eq. (23) vanishes upon inverse Fourier transforming because it has odd parity in $\mathbf{k}$ after the $\mathbf{k}'$ integral. Further manipulations allow one to write the remaining expression for the collisional current in the "Landau" form[19]

$$\mathbf{J}_v = \int d^3v' \overleftrightarrow{Q}(\mathbf{v},\mathbf{v}')\cdot\left(\frac{1}{m_{s'}}\frac{\partial}{\partial\mathbf{v}'} - \frac{1}{m_s}\frac{\partial}{\partial\mathbf{v}}\right) f_s(\mathbf{v}) f_{s'}(\mathbf{v}'), \tag{24}$$

where $\overleftrightarrow{Q}$ is the tensor kernel

$$\overleftrightarrow{Q}(\mathbf{v},\mathbf{v}') \equiv \frac{(4\pi)^2 q_s^2 q_{s'}^2}{m_s}\int \frac{d^3k}{(2\pi)^3}\frac{-i\mathbf{k}\mathbf{k}}{k^4} p_1(\mathbf{k})p_2(\mathbf{k}), \tag{25}$$

in which p_1 and p_2 are defined by

$$p_1(\mathbf{k}) \equiv \int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\hat{\varepsilon}(\mathbf{k},\omega)(\omega - \mathbf{k}\cdot\mathbf{v})(\omega - \mathbf{k}\cdot\mathbf{v}')} \tag{26}$$

and

$$p_2(\mathbf{k}) \equiv \int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{d\omega'}{2\pi}\frac{\omega' e^{-i\omega' t}}{\hat{\varepsilon}(-\mathbf{k},\omega')(\omega' + \mathbf{k}\cdot\mathbf{v})(\omega' + \mathbf{k}\cdot\mathbf{v}')}. \tag{27}$$

We have also introduced notation denoting the field particle species explicitly with s'. Writing $\mathbf{J}_v$ in the Landau form of Eq. (24) will be convenient for illuminating the physics embedded in the collision operator as well as for proving important physical properties of the collision operator in Sec. IV.

The integrals in p_1 and p_2 can be evaluated along the Landau contour using Cauchy's integral theorem to give

$$p_1 = i\left\{ \sum_j \frac{e^{-i\omega_j t}}{\left.\frac{\partial \hat{\varepsilon}(\mathbf{k},\omega)}{\partial \omega}\right|_{\omega_j} (\omega_j - \mathbf{k}\cdot\mathbf{v})(\omega_j - \mathbf{k}\cdot\mathbf{v}')} - \frac{i\pi\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]}{\hat{\varepsilon}(\mathbf{k},\mathbf{k}\cdot\mathbf{v})} e^{-i\mathbf{k}\cdot\mathbf{v}' t} \right\}, \quad (28)$$

where j denotes each mode, i.e., the dispersion relations, which are the roots of the dielectric function $\hat{\varepsilon}(\mathbf{k},\omega)=0$ from Eq. (11). In Eq. (28) we have combined the last two terms which come from the inverse Laplace transform by using the fact that $\exp[-i\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')t]$ is rapidly oscillating for large t except at $\mathbf{v}=\mathbf{v}'$, so $\hat{\varepsilon}(\mathbf{k},\mathbf{k}\cdot\mathbf{v}') \approx \hat{\varepsilon}(\mathbf{k},\mathbf{k}\cdot\mathbf{v})$. Furthermore, we have identified the relation

$$-\frac{e^{-i\mathbf{k}\cdot\mathbf{v}' t}}{\hat{\varepsilon}(\mathbf{k},\mathbf{k}\cdot\mathbf{v}')}\left[\frac{1-e^{-i\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')t}}{\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')}\right] \approx -\frac{i\pi\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]}{\hat{\varepsilon}(\mathbf{k},\mathbf{k}\cdot\mathbf{v})} e^{-i\mathbf{k}\cdot\mathbf{v}' t}, \quad (29)$$

where the Dirac delta function definition is strictly correct only in the limit $t\to\infty$. By similar arguments, Eq. (27) becomes

$$p_2 = i\left\{ \sum_j \frac{\omega_j' e^{-i\omega_j' t}}{\left.\frac{\partial \hat{\varepsilon}(-\mathbf{k},\omega')}{\partial \omega'}\right|_{\omega_j'} (\omega_j' + \mathbf{k}\cdot\mathbf{v})(\omega_j' + \mathbf{k}\cdot\mathbf{v}')} + \frac{e^{i\mathbf{k}\cdot\mathbf{v}t}}{\hat{\varepsilon}(-\mathbf{k},-\mathbf{k}\cdot\mathbf{v})} + \frac{i\pi\mathbf{k}\cdot\mathbf{v}'\,\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]e^{i\mathbf{k}\cdot\mathbf{v}t}}{\hat{\varepsilon}(-\mathbf{k},-\mathbf{k}\cdot\mathbf{v})} \right\}, \quad (30)$$

where ω_j' solves $\hat{\varepsilon}(-\mathbf{k},\omega')=0$.

Putting the product of Eqs. (28) and (30) into Eq. (25) gives an integral expression with six terms in the integrand. One term, which is the product of the last terms from Eqs. (28) and (30), is an odd function of $\mathbf{k}$ and therefore vanishes after integration. Three of the terms are rapidly oscillating in time $\sim\exp(\pm i\mathbf{k}\cdot\mathbf{v}t)$ and provide negligible contributions after integration compared to the remaining two terms which survive. We are then left with the following expression:

$$\overleftrightarrow{Q} = \frac{2q_s^2 q_{s'}^2}{m_s}\int d^3k \frac{\mathbf{k}\mathbf{k}}{k^4}\left\{ \frac{\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]}{\hat{\varepsilon}(\mathbf{k},\mathbf{k}\cdot\mathbf{v})\hat{\varepsilon}(-\mathbf{k},-\mathbf{k}\cdot\mathbf{v})} + i\sum_j \frac{\omega_j' e^{-i\omega_j' t}}{\left.\frac{\partial \hat{\varepsilon}(-\mathbf{k},\omega')}{\partial \omega'}\right|_{\omega_j'} (\omega_j' + \mathbf{k}\cdot\mathbf{v})(\omega_j' + \mathbf{k}\cdot\mathbf{v}')} \times \frac{e^{-i\omega_j t}}{\left.\frac{\partial \hat{\varepsilon}(\mathbf{k},\omega)}{\partial \omega}\right|_{\omega_j} (\omega_j - \mathbf{k}\cdot\mathbf{v})(\omega_j - \mathbf{k}\cdot\mathbf{v}')} \right\}. \quad (31)$$

Equation (31) can be further simplified by applying the reality conditions, $\hat{\varepsilon}(-\mathbf{k},-\mathbf{k}\cdot\mathbf{v})=\hat{\varepsilon}^*(\mathbf{k},\mathbf{k}\cdot\mathbf{v})$, where $*$ denotes the complex conjugate, and $\omega_j=\omega_{R,j}+i\gamma_j$ (where $\omega_{R,j}$ and γ_j are the real and imaginary parts of the jth solution of the dispersion relation) obey the properties that $\omega_{R,j}$ is an odd function of $\mathbf{k}$ while γ_j is an even function of $\mathbf{k}$. It follows then that $\omega_j'=-\omega_j^*$, and

$$\left.\frac{\partial \hat{\varepsilon}(-\mathbf{k},\omega')}{\partial \omega'}\right|_{\omega_j'} = -\left.\frac{\partial \hat{\varepsilon}^*(\mathbf{k},\omega)}{\partial \omega}\right|_{\omega_j}. \quad (32)$$

Writing ω_j' in terms of its real and imaginary parts in the last term of Eq. (31), the real part is then odd in $\mathbf{k}$ and hence vanishes upon integrating. The second term in Eq. (31) is thus only due to the imaginary part and can be written as

$$\sum_j \frac{e^{2\gamma_j t}}{\gamma_j\left|\left.\frac{\partial \hat{\varepsilon}(\mathbf{k},\omega)}{\partial \omega}\right|_{\omega_j}\right|^2}\left[\frac{\gamma_j}{(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v})^2+\gamma_j^2}\right] \times \left[\frac{\gamma_j}{(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v}')^2+\gamma_j^2}\right]. \quad (33)$$

For $\gamma_j \ll |\omega_{R,j}-\mathbf{k}\cdot\mathbf{v}|$ we can approximate the expressions in square brackets using the Lorentzian representation for a Dirac delta function

$$\frac{\Delta}{x^2+\Delta^2} \approx \pi\delta(x) \quad \text{if } \frac{\Delta}{x} \ll 1. \quad (34)$$

Thus, we find a compact expression for $\overleftrightarrow{Q}$ that captures the physics of both stable and unstable plasmas in a finite space-time domain,

$$\overleftrightarrow{Q} = \frac{2q_s^2 q_{s'}^2}{m_s}\int d^3k \frac{\mathbf{k}\mathbf{k}}{k^4}\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]\left[\frac{1}{|\hat{\varepsilon}(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2} + \sum_j \frac{\pi\delta(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v})e^{2\gamma_j t}}{\gamma_j\left|\left.\frac{\partial \hat{\varepsilon}(\mathbf{k},\omega)}{\partial \omega}\right|_{\omega_j}\right|^2}\right]. \quad (35)$$

The first term is the conventional Lenard–Balescu term that describes Coulomb interactions between particles in the plasma that are Debye shielded due to the plasma polarization. The second term represents a longer range interaction that arises from the nature of the plasma dielectric and is due to unstable modes that are excited from the thermal fluctuations of individual particles interacting with the Coulomb fields of other charged particles. In the stable plasma limit, $\gamma_j < 0$, the wave-particle interaction term rapidly decays and is entirely negligible, thus returning the Lenard–Balescu equation. The linear plasma kinetic equation is then $df_s/dt = \Sigma_{s'} C(f_s, f_{s'})$, where

$$C(f_s,f_{s'}) = -\frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v' \overset{\leftrightarrow}{Q} \cdot \left(\frac{1}{m_{s'}} \frac{\partial}{\partial \mathbf{v}'} - \frac{1}{m_s} \frac{\partial}{\partial \mathbf{v}} \right) f_s(\mathbf{v}) f_{s'}(\mathbf{v}'), \tag{36}$$

with $\overset{\leftrightarrow}{Q}$ given by Eq. (35). However, the $\exp(2\gamma_j t)$ term needs further consideration in the context of unstable plasmas.

The small $\mathbf{k}$ integration limit in Eq. (35) is resolved by accounting for plasma polarization and is a main result of the generalization of Landau's collision operator provided by Lenard and Balescu's form. However, the integral logarithmically diverges for the first term in the large $\mathbf{k}$ limit because we have not properly accounted for large-angle scattering when two point particles are in very close proximity to one another. To resolve this limit, the integral is typically cutoff at $1/b_{\min}$, where $b_{\min}$ is the minimum impact parameter, cf. Ref. 14. The same cutoff is appropriate for this term whether the plasma is stable or unstable because in either case it describes the interaction between individual particles which is limited in closeness by $b_{\min}$. The second term describing wave-particle interactions does not diverge in either the large or small $\mathbf{k}$ limit, so no cutoff is required. Additionally, wave damping mechanisms may exist for large $\mathbf{k}$ that effectively truncate the upper limit of integration.

An alternative, but equivalent, form for the kernel $\overset{\leftrightarrow}{Q}$ in Eq. (35) is

$$\overset{\leftrightarrow}{Q} = \frac{q_s^2}{m_s} \int \frac{d^3k}{(2\pi)^3} |\delta_i \tilde{\mathbf{E}}(\mathbf{k},t)|^2 \delta[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')], \tag{37}$$

where $\delta_i \tilde{\mathbf{E}}$ is the inverse Laplace transform of Eq. (16). The equivalence of Eqs. (35) and (37) can be checked by an analysis similar to that above, including the neglect of rapidly oscillating "cross" terms in $\mathbf{k}$ space, but without the ensemble average. This alternative form for $\overset{\leftrightarrow}{Q}$ shows explicitly that it is the "discrete particle" electric fields around individual particles that causes scattering. When instabilities are not present, these fields are the usual Coulomb fields of the charged particles, which are Debye shielded due to plasma polarization. In this case, scattering is effectively limited to particles within a Debye sphere of each other. The presence of instabilities, however, gives rise to a longer range interaction between particles mediated by waves excited through the plasma dielectric. In this manner scattering between two particles can reach well beyond a Debye sphere.

B. Interpretation of $e^{2\gamma t}$

A proper interpretation of time, t, in Eq. (35) requires consideration of the nature of instabilities present in the plasma. If the instabilities are absolute, i.e., the modes grow continually in time at a fixed spatial location with a vanishing group velocity, one can consider the collision operator at a fixed location to be dependent on time and hence f would evolve in time at that location. For example, if an absolute instability were to be turned on at some time t_o, the time t in Eq. (35) would simply refer to the progression of time, at some location, after the instability is present. In this case the above analysis will hold only for a few growth times, $\tau \sim 1/\gamma$, before nonlinear effects become important. The linear theory, however, would be valid for the short time scale evolution of plasmas with absolute instabilities.

Convective modes, on the other hand, propagate through the plasma with a finite group velocity $\mathbf{v}_g$. For these instabilities the fluctuation level at a fixed location in space does not grow or decay in time; rather, the waves grow as they propagate and thus the fluctuation amplitude changes for different spatial locations. In this case, time t in Eq. (35) is the time it takes a growing mode to travel from its origin to the spatial location of interest. For convective modes, f does not evolve in time at a fixed spatial location; rather, it evolves in space along the direction of propagation of the convective modes. Time is thus interpreted in a frame of reference moving with the group velocity of the wave to yield

$$2\gamma t = 2 \int_{\mathbf{x}_o(\mathbf{k})}^{\mathbf{x}} d\mathbf{x}' \cdot \frac{\mathbf{v}_g \gamma}{|\mathbf{v}_g|^2} \tag{38}$$

in which $\mathbf{x}_o(\mathbf{k})$ is the location where the mode with wavenumber $\mathbf{k}$ becomes unstable, $\mathbf{x}$ is the measurement location, and $\mathbf{x}'$ is the path between $\mathbf{x}_o$ and $\mathbf{x}$ that the mode with wavenumber $\mathbf{k}$ follows.

For convective instabilities in a homogeneous finite domain, the same $\mathbf{k}$ are unstable throughout the region since the plasma dielectric function is uniform. In this case the coordinate system can be chosen such that $\mathbf{x}_o(\mathbf{k}) = 0$. However, if small anisotropies are present, or if scattering by either the convective instabilities or Coulomb interactions alters the plasma dielectric, different $\mathbf{k}$ may be unstable at different locations in the domain and care has to be taken in determining the spatial integration limits.

III. VALIDITY OF THE LINEAR MODEL

To estimate the domain length over which instabilities can grow (either in time for absolute instabilities or space for convective instabilities) before nonlinear effects become important we need to consider Eq. (5). A conservative estimate for the maximum domain length can be obtained by considering just a single term from the left-hand side which implies the requirement $|\delta f|/f \lesssim 1$. Equations (13) and (15) lead to the scaling relationship

$$\delta\hat{f} \sim \frac{q}{m}\frac{\mathbf{k}\cdot\partial f/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}}\delta\hat{\phi}. \quad (39)$$

Typically for electrons $\omega-\mathbf{k}\cdot\mathbf{v}\sim kv_{Te}$ for a characteristic thermal electron speed v_{Te}. Using this, one finds the validity condition for the linear model reduces to the intuitive requirement that the Coulombic potential energy level of the fluctuations cannot exceed the ambient thermal energy of the plasma, $q\delta\phi/T_e \lesssim 1$. The more general condition is

$$\frac{4\pi e^2}{m}\frac{1}{f}\frac{\partial f}{\partial\mathbf{v}}\cdot\int d^3k\frac{\mathbf{k}}{k^2}\left\{\frac{\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]}{\hat{\varepsilon}(\mathbf{k},\mathbf{k}\cdot\mathbf{v})} + \sum_j \frac{e^{\gamma_j t}}{(\omega_j-\mathbf{k}\cdot\mathbf{v})(\omega_j-\mathbf{k}\cdot\mathbf{v}')\left.\frac{\partial\hat{\varepsilon}}{\partial\omega}\right|_{\omega_j}}\right\} \lesssim 1. \quad (40)$$

It is difficult to extract any more information from Eq. (40) without specifying the nature of particular instabilities. A specific example is evaluated in Sec. V. To check that Eq. (40) is consistent with that of previous models when the plasma is stable, consider a typical illustrative example of the Lenard–Balescu equation when Debye shielding is included, $\hat{\varepsilon}=1+1/k^2\lambda_{\mathrm{De}}^2$. In this case, Eq. (40) reduces to $\ln\Lambda/n\lambda_{\mathrm{De}}^3 \lesssim 1$, which is consistent with previous analyses which used the BBGKY hierarchy to justify the derivation of the Lenard–Balescu equation using the test particle method.[20] Here $\Lambda\equiv\lambda_D/b_{\min}$, where $b_{\min}$ is the minimum impact parameter.

IV. PHYSICAL PROPERTIES OF THE COLLISION OPERATOR

A physically meaningful collision operator should obey certain properties, such as, conservation laws and the Boltzmann $\mathcal{H}$-theorem. In particular, Eq. (36) obeys properties (a)–(e) below. The important features of the collision operator required to prove properties (a)–(c) are that it can be written in the Landau form of Eq. (24) and that

$$\overleftrightarrow{Q}\cdot(\mathbf{v}-\mathbf{v}')=0, \quad (41)$$

which follows from the fact that $\overleftrightarrow{Q}(\mathbf{v},\mathbf{v}')=\overleftrightarrow{Q}(\mathbf{v}',\mathbf{v})$. Since $\overleftrightarrow{Q}$ exhibits these features, proof of properties (a)–(c) for Eq. (35) are identical to those for the Lenard–Balescu equation provided by Lenard in his original work.[1] Therefore, we omit the detailed proofs here.

(a) *Conservation of particle density, momentum, and energy*. Conservation of density is simply a statement that collisions, whether through interaction with waves or other particles, do not create or destroy particles. This statement is described by

$$\int d^3v C(f_s)=0. \quad (42)$$

Conservation of momentum,

$$\int d^3v m_s\mathbf{v}C(f_s,f_{s'}) + \int d^3v m_{s'}\mathbf{v}C(f_{s'},f_s)=0, \quad (43)$$

states that momentum lost by one species is gained by the other species. Coulomb electric fields and waves are each an intermediary for the momentum transfer. Equation (43) also implies that momentum is conserved for collisions within a species. A conservation of energy property is also present,

$$\int d^3v\frac{1}{2}m_s v^2 C(f_s,f_{s'}) + \int d^3v\frac{1}{2}m_{s'}v^2 C(f_{s'},f_s)=0. \quad (44)$$

(b) *If $f\geqslant 0$ initially, then $f\geqslant 0$ always*. This property guarantees that f stays physically meaningful throughout the time evolution.

(c) *The Maxwellian is an equilibrium for any species*. The Maxwellian distribution with flow $f_M(\mathbf{v}) = \exp(-Av^2/2+\mathbf{B}\cdot\mathbf{v}+C)$, in which $A>0$, $\mathbf{B}$ and C are constants, satisfies $C(f_M,f_M)=0$. Therefore it is a stationary, i.e., equilibrium, solution.

(d) *$C(f)$ is Galilean invariant*. *Proof:* Transforming coordinates to $\mathbf{w}\equiv\mathbf{v}-\mathbf{V_f}$ and $\mathbf{w}'\equiv\mathbf{v}'-\mathbf{V_f}$ changes the characteristic equations to $\mathbf{x}'=\mathbf{x}+(\mathbf{w}+\mathbf{V}_f)(t'-t)$ and $\mathbf{w}'=\mathbf{w}$ with the same initial conditions $\mathbf{x}'(t'=t)=\mathbf{x}$ and $\mathbf{w}'(t'=t)=\mathbf{w}$. This introduces a Doppler shift into Eqs. (9)–(11), where $\omega\leftrightarrow\omega+\mathbf{k}\cdot\mathbf{V_f}$ when $\mathbf{v}\leftrightarrow\mathbf{w}$. By defining the variables

$$\bar{\omega}\equiv\omega+\mathbf{k}\cdot\mathbf{V}_f \quad \text{and } \bar{\omega}'\equiv\omega'+\mathbf{k}\cdot\mathbf{V}_f \quad (45)$$

which satisfy $d^3\omega=d^3\bar{\omega}$ and $\partial/\partial\bar{\omega}=\partial/\partial\omega$, we can replace ω with $\bar{\omega}$, ω' with $\bar{\omega}'$, $\mathbf{v}$ with $\mathbf{w}$, $\mathbf{v}'$ with $\mathbf{w}'$ and the entire analysis of Sec. II A can be repeated in these new coordinates. Thus, the collision operator, Eq. (36), is Galilean invariant. Therefore the kernel satisfies the Galilean invariance, $Q(\mathbf{v},\mathbf{v}')=Q(\mathbf{w},\mathbf{w}')$, as well.

(e) *Boltzmann $\mathcal{H}$-theorem*.[21] The $\mathcal{H}$-functional,

$$\mathcal{H}\equiv\int d^3v f(\mathbf{v})\ln f(\mathbf{v}) \quad (46)$$

satisfies $d\mathcal{H}/dt\leqslant 0$ for any distribution function $f(\mathbf{v})$. Furthermore, as $t\to\infty$, $f(\mathbf{v})$ always tends to the Maxwellian $f_M = \exp(-Av^2/2+\mathbf{B}\cdot\mathbf{v}+C)$.

Proof: From the definition of $\mathcal{H}$,

$$\frac{d\mathcal{H}}{dt}=\int d^3v[1+\ln f]\frac{\partial f}{\partial t}. \quad (47)$$

Putting in $df/dt=C(f_s)$, where $C(f_s)$ is given in Eq. (36), then adding it to the form of the same expression with $\mathbf{v}$ switched with $\mathbf{v}'$ gives the quadratic form

$$2\frac{d\mathcal{H}}{dt}=-\frac{1}{m_s}\int d^3v\int d^3v'\mathbf{X}\cdot\overleftrightarrow{Q}(\mathbf{v},\mathbf{v}')\cdot\mathbf{X}, \quad (48)$$

where

$$\mathbf{X}\equiv\frac{\partial\ln f_s(\mathbf{v}')}{\partial\mathbf{v}'}-\frac{\partial\ln f_s(\mathbf{v})}{\partial\mathbf{v}}. \quad (49)$$

Using Eq. (35) for $\overleftrightarrow{Q}$, writing $\delta(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v})$ in the form given in Eq. (34) and performing one $\mathbf{k}$ integral yields

$$\mathbf{X}\cdot\overleftrightarrow{Q}\cdot\mathbf{X} = \frac{2q_s^4}{m_s}\int d^2k \frac{(\mathbf{k}\cdot\mathbf{X})^2}{k^4|\mathbf{v}-\mathbf{v}'|}\left\{\frac{1}{|\hat{\varepsilon}(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2} + \sum_j \frac{e^{2\gamma_j t}}{\left|\frac{\partial\hat{\varepsilon}(\mathbf{k},\omega)}{\partial\omega}\right|_{\omega_j}^2[(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v})^2+\gamma_j^2]}\right\}, \tag{50}$$

which is positive for all $\mathbf{v}$ and $\mathbf{v}'$. Thus, one finds that

$$\frac{d\mathcal{H}}{dt} \leqslant 0 \tag{51}$$

for all f. Equality, i.e., $d\mathcal{H}/dt=0$, only holds when $\mathbf{X}$ is parallel to the vector $\mathbf{v}-\mathbf{v}'$ which implies that f must have the general Maxwellian form

$$f(\mathbf{v}) = f_M(\mathbf{v}) = \exp(-1/2Av^2 + \mathbf{B}\cdot\mathbf{v} + C), \tag{52}$$

where the five constants A, $\mathbf{B}$, and C are determined by the initial conditions. Proof of this final step is provided by Lenard.[1]

The $\mathcal{H}$-theorem shows that the plasma evolves toward a Maxwellian equilibrium not only by Coulomb scattering between individual particles, but also by wave-particle scattering due to instabilities in a finite space-time domain. The time scale on which equilibrium is realized is set by the scattering rate which, for unstable plasmas, is often dominated by wave-particle interactions. That is to say, a non-Maxwellian distribution will become Maxwellian much more rapidly in the presence of instabilities, for example, convective instabilities, than it would otherwise. A simple illustrative example is considered in the next section.

V. APPLICATION TO THE ION-ACOUSTIC INSTABILITY

To illustrate enhanced scattering by convective modes we consider the ion-acoustic instability excited by a Maxwellian ion distribution flowing relative to a stationary Maxwellian electron distribution with fluid flow velocity $\mathbf{u}$. For flowing Maxwellian distributions the dielectric function, Eq. (11) reduces to

$$\hat{\varepsilon}(\mathbf{k},\omega) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2 v_{Ts}^2} Z'(\xi_s), \tag{53}$$

where $\omega_{ps} \equiv \sqrt{4\pi n_s q_s^2/m_s}$ is the plasma frequency of species s, Z' is the derivative of the plasma dispersion function with respect to ξ_s,

$$\xi_s \equiv \frac{\omega - \mathbf{k}\cdot\mathbf{u}_s}{kv_{Ts}}, \tag{54}$$

and $k \equiv |\mathbf{k}|$.

In this example we assume that $\mathbf{u}_e = 0$ and $T_e \gg T_i$. For these parameters, the ion-acoustic instability is excited. Furthermore, we assume that $\xi_e \ll 1$ and $\xi_i \gg 1$. In this case the plasma dispersion function can be expanded in a power series for electrons and asymptotically for ions to yield

$$\hat{\varepsilon}(\mathbf{k},\omega) = 1 + \frac{1}{k^2\lambda_{\rm De}^2} - \frac{\omega_{pi}^2}{(\omega-\mathbf{k}\cdot\mathbf{u})^2} + i\frac{\sqrt{\pi}}{k^2\lambda_{\rm De}^2}\frac{\omega}{kv_{Te}}. \tag{55}$$

For this example, we have assured that ion Landau damping is negligible due to the cold ion approximation $T_e \gg T_i$. Solving for the dispersion relation, $\hat{\varepsilon}=0$ from Eq. (55), reveals the two solutions

$$\omega_{\pm} = \left(\mathbf{k}\cdot\mathbf{u} \pm \frac{kc_s}{\sqrt{1+k^2\lambda_{\rm De}^2}}\right) \times\left[1 \mp i\sqrt{\frac{\pi m_e}{8M_i}}\frac{1}{(1+k^2\lambda_{\rm De}^2)^{3/2}}\right], \tag{56}$$

where $c_s^2 = T_e/M_i$ is the sound speed. At most, only one unstable mode is present for any particular wavenumber $\mathbf{k}$, and the other mode is stable for that $\mathbf{k}$. The group velocity of the unstable mode is in the ion flow direction and the instability criterion for the flow speed is given by

$$|\mathbf{k}\cdot\mathbf{u}| > \frac{kc_s}{\sqrt{1+k^2\lambda_{\rm De}^2}}. \tag{57}$$

The real and imaginary parts of the dispersion relation, Eq. (56), are plotted as a function of $k\lambda_{\rm De}$ in Fig. 1 for a particular $\mathbf{k}\cdot\mathbf{u}$. In the figures, we assume that the ion fluid flow is a nonzero constant, in particular, $u \sim 2c_s$ over a finite length L and is zero elsewhere, as illustrated in Fig. 2. We then estimate the collision frequency in the unstable region as compared to the stable region and answer the following questions: (1) What is the minimum length $L_{\min}$, where wave-particle scattering dominates particle-particle scattering in the unstable region? (2) What is the maximum length $L_{\max}$ where nonlinear effects become important and our analysis is superseded by a nonlinear one?

The collision frequency for electron-ion scattering, $v \sim v_{Te}$ and $v' \sim v_{Ti}$, can be estimated from the plasma kinetic equation $df/dt = C(f) \Rightarrow \nu \sim C(f)/f$. From Eq. (36), we estimate for electrons

$$\nu \sim \frac{n}{m_e v_{Te}^2} Q, \tag{58}$$

where we estimate Q as a scalar. To highlight the different mechanisms for scattering we split Q into two terms $Q \equiv Q_{\rm LB} + Q_w$, where $Q_{\rm LB}$ is the Lenard–Balescu kernel due to Coulomb scattering between particles and Q_w is due to wave-particle scattering from the unstable part of k-space.

To estimate the Lenard–Balescu term requires the plasma dielectric evaluated at $\mathbf{k}\cdot\mathbf{v} \sim kv_{Te}$, which leads to

$$|\hat{\varepsilon}(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2 \approx \left(1 + \frac{1}{k^2\lambda_{\rm De}^2}\right)^2 + \frac{\pi}{k^4\lambda_{\rm De}^4}, \tag{59}$$

where we have used $kv_{Te} \gg \omega_{pi}$. Using this in the k-space integral for $Q_{\rm LB}$ and integrating from $k=0$ to $k=1/b_{\min}$, where $b_{\min}$ is the minimum impact parameter, gives the familiar result

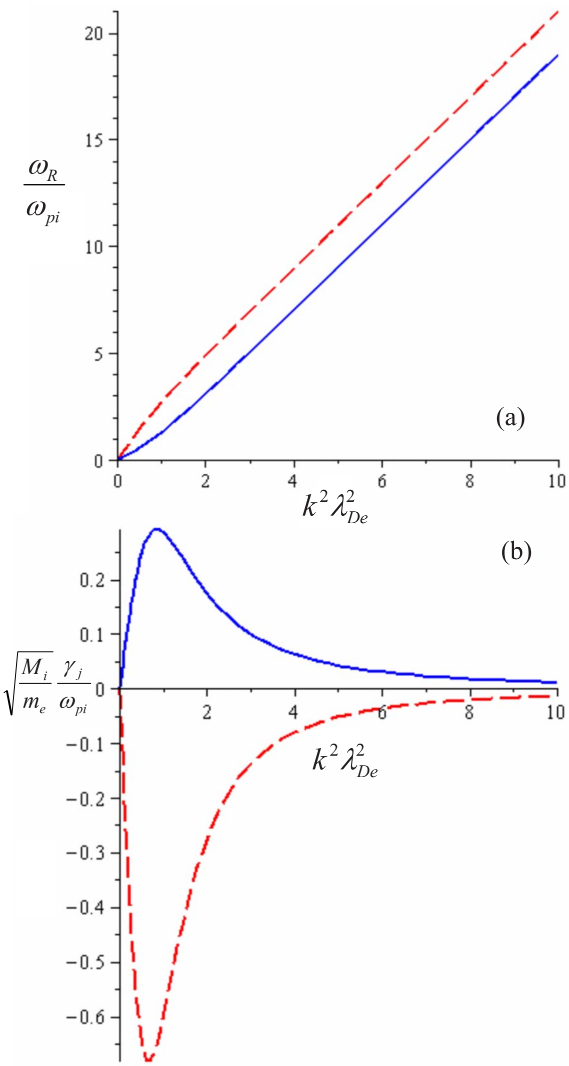

FIG. 1. (Color online) (a) Real part of the frequency determined from the ion-acoustic dispersion relation normalized to the ion plasma frequency as a function of $k\lambda_{\rm De}$ with $\mathbf{k}\cdot\mathbf{u}/k=2$. (b) Corresponding imaginary parts. In this case ω_+ (red, dashed line) is damped while ω_- (blue, solid line) is unstable. If the sign of $\mathbf{k}$ is changed ω_+ grows while ω_- damps since $\omega_+(-\mathbf{k})=-\omega_-(\mathbf{k})$.

$$Q_{\rm LB} \sim \frac{2\pi e^4}{m_e v_{Te}} \ln \Lambda \tag{60}$$

in which $\Lambda \equiv \lambda_{\rm De}/b_{\rm min} \gg 1$.

An estimate for Q_w requires the scaling

$$\left|\frac{\partial\hat{\varepsilon}}{\partial\omega}\right|^2_{\omega_R} \approx 4\frac{(1+k^2\lambda_{\rm De}^2)^3}{k^6\lambda_{\rm De}^4 c_s^2}, \tag{61}$$

where we have neglected the higher order term resulting from the imaginary part of $\hat{\varepsilon}$. Using this along with $\mathbf{k}\cdot\mathbf{v} \sim kv_{Te} \gg \omega_R$ leads to the estimate, using $w\equiv k\lambda_{\rm De}$,

$$Q_w \sim \frac{\pi e^4}{m_e v_{Te}}\int_0^\infty dw \frac{w^3}{(1+w^2)^3}\exp(2\gamma t). \tag{62}$$

The exponent from Eq. (38) reduces to $2\gamma t \approx \eta w/(1+w^2)^{3/2}$, where

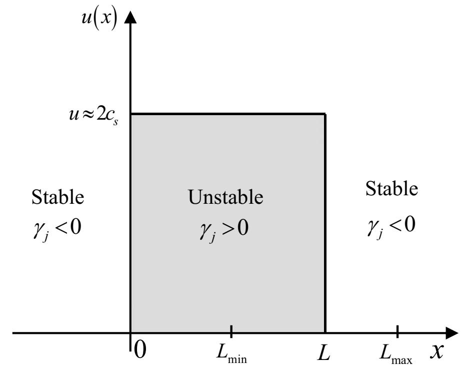

FIG. 2. Schematic drawing of the example problem. A finite ion flow of $u \sim 2c_s$ is imposed over a finite region L of the otherwise stable plasma.

$$\eta \equiv \sqrt{\frac{\pi m_e}{2M_i}}\frac{x}{\lambda_{\rm De}} \tag{63}$$

for this example as long as $u\sim c_s$.

Upon estimating the integral in Eq. (62) for $\eta \gtrsim 1$, we find

$$Q_w \sim \frac{Q_{\rm LB}}{\ln\Lambda}\frac{1}{8}\frac{\sqrt{\pi}}{\sqrt{\eta}}e^{\eta/\sqrt{8}}. \tag{64}$$

Smaller values of η do not contribute to enhanced scattering since $Q_w \ll Q_{\rm LB}$ for this case. Equation (64) shows that wave-particle scattering becomes dominant when

$$\frac{1}{8\ln\Lambda}\frac{\sqrt{\pi}}{\sqrt{\eta}}e^{\eta/\sqrt{8}} \gtrsim 1, \tag{65}$$

which for $\ln\Lambda \approx 10$ implies $\eta \gtrsim 14$ so wave-particle scattering dominates for $L > L_{\rm min}$, where

$$L_{\rm min} \approx 28\sqrt{\frac{2M_i}{\pi m_e}}\lambda_{\rm De}. \tag{66}$$

Having answered question (1), we now seek to determine the distance $L_{\rm max}$ beyond which nonlinear effects become important. Equation (40) for this example reduces to

$$\frac{1}{n\lambda_{\rm De}^3}\int_0^\infty dw \frac{w}{(1+w^2)^{3/2}}\sum_j e^{\gamma_j t} \lesssim 1, \tag{67}$$

where we have used $\omega_j - \mathbf{k}\cdot\mathbf{v} \sim kv_{Te}$ and $\omega_j - \mathbf{k}\cdot\mathbf{v}' \sim kc_s$. Using the definition in Eq. (63) and evaluating the integral for $\eta \gg 1$ in Eq. (67) yields the condition

$$\frac{1}{n\lambda_{\rm De}^3}\sqrt{\frac{8\pi}{\eta}}e^{\eta/4\sqrt{2}} \lesssim 1, \tag{68}$$

which determines the space domain in which the linear theory can be applied to our example. Conditions (65) and (68) can be solved using the Lambert W-function to show that wave-particle scattering dominates and the linear theory is valid when

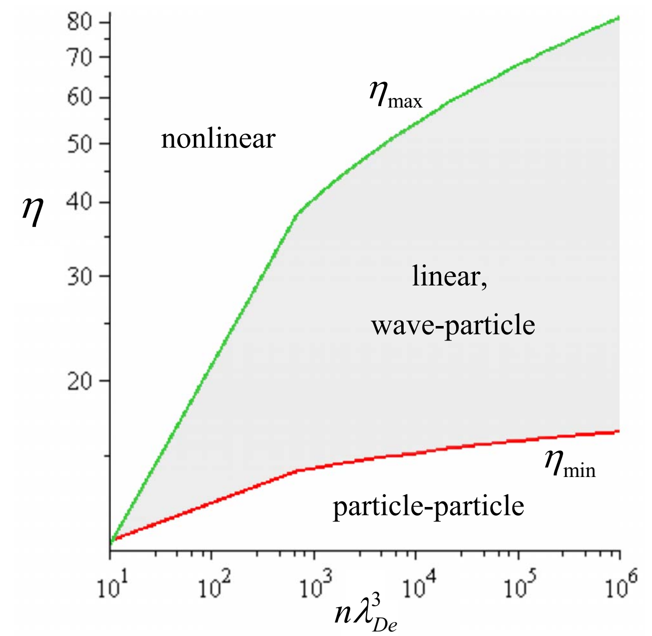

FIG. 3. (Color online) Plot of the maximum dimensionless distance η_{max} such that the linear analysis is valid ($\eta<\eta_{max}$) and the minimum dimensionless distance η_{min} such that wave-particle interactions dominate the conventional particle-particle scattering. For $\eta>\eta_{max}$, nonlinear effects may be significant.

$$-\sqrt{2}W\left[-\frac{\pi}{64\sqrt{2}(\ln\Lambda)^2}\right]\lesssim\eta\lesssim-2\sqrt{2}W\left[-\frac{4\pi}{\sqrt{2}(n\lambda_{\mathrm{De}}^3)^2}\right]. \tag{69}$$

A plot of the conditions in Eq. (69), which shows the parameter space in which both the linear theory is valid and wave-particle scattering dominates, is shown in Fig. 3 for five decades in the plasma parameter $n\lambda_{\mathrm{De}}^3$. In Fig. 3 we have taken $\ln\Lambda\approx\ln(n\lambda_{\mathrm{De}}^3)$.

As a concrete example, if $n\lambda_{\mathrm{De}}^3\sim10^3$ for an electron-proton plasma, then $L_{\max}\approx10^3\lambda_{\mathrm{De}}$. A typical Debye length for a low-temperature laboratory plasma may be $\lambda_{\mathrm{De}}\sim10^{-2}$ cm. Thus, the unstable region where the linear wave-particle scattering can be dominant and applicable (i.e., not nonlinear) can be up to about 10 cm in length for this case.

The effective collision frequency in terms of η can thus be written

$$\nu_{\mathrm{eff}}\sim\nu_o\left(1+\frac{1}{8\ln\Lambda}\sqrt{\frac{\pi}{\eta}}e^{\eta/\sqrt{8}}\right) \tag{70}$$

which illustrates each mechanism for particle scattering and in which

$$\nu_o\equiv\frac{\omega_{pe}}{8\pi n\lambda_{\mathrm{De}}^3}\ln\Lambda \tag{71}$$

is a reference collision frequency. A plot of each source of scattering as a function of the dimensionless position η is shown in Fig. 4. It is apparent that wave-particle scattering dominates in the unstable region of the plasma.

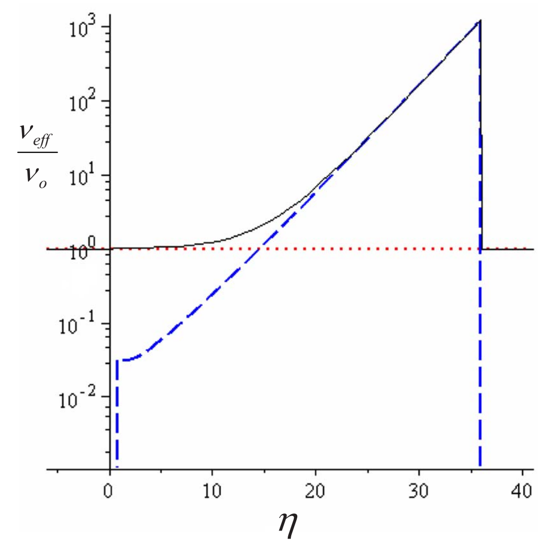

FIG. 4. (Color online) The effective collision frequency, ν_{eff}, normalized to the reference collision frequency, ν_o, for Coulomb collisions, i.e., the Lenard–Balescu collision operator (red, dotted line), due to wave-particle interactions (blue, dashed line) and the total (black, solid line). Here the ion flow, $u\approx2c_s$, is present for $0<\eta<35$ and is zero otherwise and we have assumed $\ln\Lambda=10$.

VI. CONCLUSION

A formalism similar to Lenard–Balescu theory for describing the collisional equilibration of plasma distribution functions is developed to include plasmas which are unstable, either convectively or absolutely, in a finite space or time domain, respectively. The collision operator, Eq. (36), is written in the "Landau" form which elucidates the physics of scattering between the test particle species s and any species s' (including self-collisions within s itself). Modifications to the collision operator which include wave-particle interactions due to plasma instabilities are contained in the kernel $\overleftrightarrow{Q}$ of Eq. (35). Wave-particle scattering acts to extend the range over which charged particles can interact to well beyond the conventional Debye sphere within which conventional Coulomb interactions are confined.

Writing the collision operator in the "Landau" form with kernel $\overleftrightarrow{Q}$ enables simple proofs that it satisfies conservation of particles, momentum, and energy as well as Galilean invariance and the Boltzmann $\mathcal{H}$-theorem. It has also been shown that the Maxwellian distribution is the only equilibrium solution. A direct consequence is that a non-Maxwellian distribution function equilibrates to a Maxwellian much more rapidly when instabilities are present than it does in a stable plasma. The theory's validity is limited to the finite space-time domain described in Eq. (40). Outside this domain nonlinear effects may become important.

The ion-acoustic instability is a convective mode which exemplifies important consequences of the linear wave-particle scattering theory. In particular, for a constant ion fluid velocity of order the sound speed relative to a stationary electron distribution, the minimum length for which wave-particle scattering dominates the conventional particle-particle scattering is $\mathcal{O}(\lambda_{\mathrm{De}}\sqrt{M_i/m_e})$. An example problem with ion-acoustic instabilities is constructed that exhibits a case that the linear particle-discreteness theory can describe, but which cannot be accurately predicted using stable plasma or nonlinear theories. In this example, linear electrostatic ion acoustic waves are excited from the thermal fluctuations inherent in the plasma. These waves effectively extend the interaction distance of individual discrete particles, which enhances the collision frequency resulting in more rapid relaxation to the Maxwellian equilibrium.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the anonymous referee for pointing out an error in our original presentation of ion-acoustic instabilities in Sec. V.

This material is based upon work supported under a National Science Foundation Graduate Research Fellowship (S.D.B.) and by the U.S. Department of Energy Grant No. DE-FG02-86ER53218.

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[16] See, for example, Sec. IX, p. 216, of Ref. 15 where this definition of the ensemble average is used to derive the Lenard–Balescu equation.
[17] That "unlike" particle terms vanish can be shown explicitly by first inverting the Laplace transforms, then using the definition of $\hat{\varepsilon}$ from Eq. (11). The $\mathbf{k}'$ integrand becomes $h(\mathbf{k}')\mathbf{k}'\delta(\mathbf{k}')$, where $h(\mathbf{k}')\to c$ in which c is a constant as $\mathbf{k}'\to 0$. Thus these terms are zero upon integrating over $\mathbf{k}'$.
[18] L. D. Landau, Phys. Z. Sowjetunion **10**, 154 (1936) [English translation in *Collected Papers of L. D. Landau*, edited by D. Ter Harr (Pergamon, London, 1965)].
[19] Here "Landau" form refers to the integrand of $\hat{\mathbf{J}}_v$ being written as a tensor kernel times an asymmetric term for the velocity derivatives of the distribution functions such as Landau did in his seminal paper, cf. Eq. (11) of Ref. 18. The difference between Landau's work, our work, and that of Lenard and Balescu lies in different definitions of the kernel $\overleftrightarrow{Q}$.
[20] See, for example, p. 211 in Sec. IX and Appendix IV of Ref. 15.
[21] The proof given here is a simple extension of the $\mathcal{H}$-theorem proof originally provided by Lenard in Ref. 1 to account for the wave-particle interaction term in $\overleftrightarrow{Q}$. It is, however, essentially the same.