Kinetic theory of instability-enhanced collisional effects[a)]

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The Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy is used to derive a generalization of the Lenard–Balescu plasma kinetic equation that accounts for wave-particle scattering due to instabilities that originate from discrete particle motion. Application to convective instabilities is emphasized for which the growing waves either propagate out of the domain of interest or modify the particle distribution to reduce the instability amplitude before nonlinear amplitudes are reached. Two such applications are discussed: Langmuir's paradox and determining the Bohm criterion for multiple ion species plasmas. In these applications, collisions are enhanced by ion-acoustic and ion-ion two-stream instabilities, respectively. The relationship between this kinetic theory and quasilinear theory is discussed.
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I. INTRODUCTION

Plasma kinetic theories typically assume either that the plasma is stable, in which case scattering is dominated by conventional Coulomb interactions between individual particles, or that fluctuations due to instabilities have grown to such large amplitudes that collective wave motion is the dominant mechanism for scattering particles. In this work, we consider an intermediate regime: weakly unstable plasmas in which collective fluctuations may be, but are not necessarily, the dominant scattering mechanism and for which the collective fluctuation amplitude is sufficiently weak that nonlinear effects are subdominant. We emphasize convective instabilities that either leave the plasma (or region of interest) or modify the particle distribution functions to limit the fluctuation amplitude before nonlinear amplitudes are reached. We discuss applications for each of these cases and show that collective fluctuations can be the dominant mechanism for scattering particles even when they are in a linear growth regime. The collision operator we discuss here was first derived in Ref. 1 using a test-particle approach. In this paper, we derive the same operator starting from a fundamentally different formalism using the Liouville equation and the BBGKY hierarchy (after Bogoliubov,[2] Born,[3] Green,[3] Kirkwood,[4] and Yvon[5]).

Kinetic equations for weakly unstable plasmas have also been developed by other authors. Frieman and Rutherford[6] used a BBGKY hierarchy approach, but focused on nonlinear aspects such as mode coupling that enters the kinetic equation at higher order in the hierarchy expansion than we consider in this work. The part of their collision operator that described collisions between particles and collective fluctuations also depended on an initial fluctuation level that must be determined external to the theory. Rogister and Oberman[7] started from a test-particle approach and focused on the linear growth regime, similar to the approach in Ref. 1, but the fluctuation-induced scattering term in their kinetic equation also depended on specifying an initial fluctuation level external to the theory. Imposing an initial fluctuation level is also a feature of Vlasov theories of fluctuation-induced scattering, such as quasilinear theory.[8,9] A distinguishing feature of this work (and Ref. 1) is that the initial fluctuation level, which subsequently becomes amplified and leads to instability-enhanced collisions, is self-consistently accounted for by its association with discrete particle motion.

Related work by Kent and Taylor[10] used the Wentzel–Kramers–Brillouin (WKB) approximation to calculate the amplification of convective fluctuations from discrete particle motion. They focused on describing the fluctuation amplitude, rather than a kinetic equation for particle scattering, and emphasized drift-wave instabilities in magnetized inhomogeneous systems. Baldwin and Callen[11] derived a kinetic equation accounting for the source of fluctuations and their effects on instability-enhanced collisional scattering in the specific case of loss-cone instabilities in magnetic mirror devices.

Our work develops a comprehensive collision operator for unmagnetized plasmas in which electrostatic instabilities that originate from discrete particle motion are present. The resultant collision operator consists of two terms. The first term is the Lenard–Balescu collision operator[12,13] that describes scattering due to the Coulomb interaction acting between individual particles. The second term is an instability-enhanced collision operator that describes scattering due to collective wave motion. Each term can be written in the Landau form,[14] which has both diffusion and drag components in velocity space. The ability to write the collision operator in the Landau form allows proof of physical properties such as the Boltzmann $\mathcal{H}$-theorem and conservation laws for collisions between individual species.

A prominent model used to describe scattering in weakly unstable plasma is quasilinear theory.[8,9] Quaslinear theory is "collisionless," being based on the Vlasov equation, but has an effective "collision operator" in the form of a diffusion

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equation that describes wave-particle interactions due to fluctuations. In the kinetic theory presented here, the instability-enhanced term of the total collision operator for species s, which is a sum of the component collision operators describing collisions of s with each species s' $C(f_s) = \Sigma_{s'} C(f_s, f_{s'})$, fits into the diffusion equation framework of quasilinear theory. This is because the drag term of the Landau form vanishes in the total collision operator (but not necessarily in the component collision operators). The instability-enhanced contribution to the total collision operator may also be considered an extension of quasilinear theory for the case that instabilities arise internal to a plasma. Conventional quasilinear theory requires specification of an initial electrostatic fluctuation level external to the theory. The kinetic prescription provides this by self-consistently accounting for the continuing source of fluctuations from discrete particle motion. This determines the spectral energy density of the plasma, which is otherwise an input parameter in conventional quasilinear theory.

This paper is organized as follows. In Sec. II, we derive a collision operator for weakly unstable plasmas using the BBGKY hierarchy. Section III provides a description of how the instability-enhanced term of this kinetic equation fits into the framework of quasilinear theory, but also provides a unique prescription for the source of fluctuations. In Sec. IV, physical properties of the resultant collision operator are reviewed. Section V describes two applications to which the kinetic equation has been applied.[15,16] The first of these is Langmuir's paradox,[17,18] which is a measurement of enhanced electron-electron scattering above the Coulomb level for a stable plasma. We consider the role of instability-enhanced collisions due to ion-acoustic instabilities in the presheath region of Langmuir's discharge and show that they significantly enhance scattering even though the instabilities propagate out of the plasma before reaching nonlinear levels.[15] The second application we consider is determining the Bohm criterion (i.e., the speed at which ions leave a plasma) in plasmas with multiple ion species. In this case we show that when ion-ion two-stream instabilities arise in the presheath they cause an instability-enhanced collisional friction that is very strong and forces the speeds of each ion species toward a common speed at the sheath-presheath boundary.[16]

II. KINETIC COLLISION OPERATOR

In Ref. 1, a kinetic equation for weakly unstable plasma was derived using a test-particle approach. A test-particle approach is based on describing the physical-space and velocity-space position of each particle individually. The distribution for all N individual particles, $F = \Sigma_i^N \delta[\mathbf{x} - \mathbf{x}_i(t)]\delta[\mathbf{v} - \mathbf{v}_i(t)]$ is a function of six phase-space dimensions plus time. It evolves according to the Klimontovich equation $dF/dt = 0$, in which $d/dt = \partial/\partial t + \mathbf{v}\cdot\partial/\partial\mathbf{x} + \mathbf{a}\cdot\partial/\partial\mathbf{v}$ is the convective derivative. A kinetic equation can then be derived by separating the smoothed and discrete particle components of F, $F = f + \delta f$, where $f \equiv \langle F\rangle$. Here $\langle\cdots\rangle$ denotes an ensemble average and $\langle\delta f\rangle = 0$. For the test-particle problem the ensemble averaging procedure is defined by the coarse graining integral $\langle\cdots\rangle \equiv (nV)^{-1}\Pi_{l=1}^N \int d^3x_{lo} d^3v_{lo} f(\mathbf{v}_{lo})(\cdots)$ in which n is the particle density and V a macroscopic volume for which $nV = N$.[19,20] The kinetic equation then follows from using the linear part of the δf evolution equation, and Gauss's law, to find the collision operator for the kinetic equation of f.

In this work, we derive an equivalent kinetic equation starting from a fundamentally different approach: the Liouville theorem and BBGKY hierarchy.[19–22] This approach starts by describing the state of the entire plasma as a single system, or point, in a 6-N-dimensional phase space that evolves in time. Letting $D_N(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_N)$ denote the probability distribution function of the system, in which $\mathbf{X}_i(t) = [\mathbf{x}_i(t), \mathbf{v}_i(t)]$, the Liouville theorem states that this distribution is constant along the path that the system follows in phase space: $D_N[\mathbf{X}_1(t=0), \mathbf{X}_2(t=0), \ldots, \mathbf{X}_N(t=0)] = D_N[\mathbf{X}_1(t), \mathbf{X}_2(t), \ldots, \mathbf{X}_N(t)]$. Taking a convective derivative gives the Liouville equation, $dD_N/dt = 0$, which can also be written in the form[21]

$$\frac{\partial D_N}{\partial t} + \sum_{i=1}^{N} \left\{ \mathbf{v}_i \cdot \frac{\partial D_N}{\partial \mathbf{x}_i} + \mathbf{a}_i \cdot \frac{\partial D_N}{\partial \mathbf{v}_i} \right\} = 0, \tag{1}$$

in which $\mathbf{v}_i \equiv \partial\mathbf{x}_i/\partial t$ and $\mathbf{a}_i \equiv \partial\mathbf{v}_i/\partial t$. Applying the Coulomb approximation, we assume no applied electric or magnetic fields, and neglect the magnetic fields produced by charged particle motion. Since we only consider forces due to the electrostatic interaction between particles, the acceleration vector can be identified as

$$\mathbf{a}_i = \sum_{j,j\neq i} \mathbf{a}_{ij}(\mathbf{x}_i - \mathbf{x}_j) = \sum_{j,j\neq i} \frac{q_i q_j}{m_i} \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}. \tag{2}$$

The reduced distributions, $f_1, f_2, \ldots, f_N$, are defined as[21]

$$f_\alpha(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_\alpha, t) \equiv N^\alpha \int d^6X_{\alpha+1} \cdots d^6X_N D_N(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_N). \tag{3}$$

Equation α of the BBGKY hierarchy of equations is formed by integrating Eq. (1) over the $6(N-\alpha)$ phase-space coordinates, $\int d^6\mathbf{X}_{\alpha+1} \cdots d^6\mathbf{X}_N$, which yields

$$\frac{\partial f_\alpha}{\partial t} + \sum_{i=1}^{\alpha} \mathbf{v}_i \cdot \frac{\partial f_\alpha}{\partial \mathbf{x}_i} + \sum_{i=1}^{\alpha}\sum_{j=1}^{\alpha} \mathbf{a}_{ij} \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}_i} + \frac{N-\alpha}{N} \sum_{i=1}^{\alpha} \int d^6\mathbf{X}_{\alpha+1} \mathbf{a}_{i,\alpha+1} \cdot \frac{\partial f_{\alpha+1}}{\partial \mathbf{v}_i} = 0. \tag{4}$$

A closure scheme is required to solve Eq. (4). We apply the standard Mayer cluster expansion[23]

$$\begin{aligned} f_1(\mathbf{X}_1) &= f(\mathbf{X}_1), \\ f_2(\mathbf{X}_1, \mathbf{X}_2) &= f(\mathbf{X}_1)f(\mathbf{X}_2) + P_{12}(\mathbf{X}_1, \mathbf{X}_2), \\ f_3(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) &= f(\mathbf{X}_1)f(\mathbf{X}_2)f(\mathbf{X}_3) + f(\mathbf{X}_1)P_{23}(\mathbf{X}_2, \mathbf{X}_3) \\ &\quad + f(\mathbf{X}_2)P_{13}(\mathbf{X}_1, \mathbf{X}_3) + f(\mathbf{X}_3)P_{12}(\mathbf{X}_1, \mathbf{X}_2) \\ &\quad + T(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3), \end{aligned} \tag{5}$$

in which f is the velocity distribution function, P is the pair correlation, and T is the triplet correlation. The closure scheme we use is to neglect the triplet correlation T. In a stable plasma, it can be shown that $T/fP \sim \mathcal{O}(\Lambda^{-1})$, where $\Lambda \sim n\lambda_D^3$.[20] In unstable plasmas, such as we consider here, the small parameter becomes Λ^{-1} times the amplification of collisions due to instabilities. After the instability amplitude becomes too large, this parameter is no longer small and T, as well as higher order terms, must be included. Some nonlinear effects, such as mode coupling, enter the hierarchy at the triplet correlation T level.[6]

Putting the expansion of Eq. (5) into Eq. (4) for $\alpha=1,2$ and neglecting T gives the two lowest order equations[21]

$$\frac{\partial f(\mathbf{X}_1)}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f(\mathbf{X}_1)}{\partial \mathbf{x}_1} + \mathbf{a}_1 \cdot \frac{\partial f(\mathbf{X}_1)}{\partial \mathbf{v}_1} = -\int d^6X_2 \mathbf{a}_{21} \cdot \frac{\partial P_{12}}{\partial \mathbf{v}_1}, \tag{6}$$

and

$$\left[\frac{\partial}{\partial t} + \sum_{i=1}^{2}\left(\mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} + \mathbf{a}_i \cdot \frac{\partial}{\partial \mathbf{v}_i} + \sum_{\substack{j=1\\ j\neq i}}^{2} \mathbf{a}_{ij} \cdot \frac{\partial}{\partial \mathbf{v}_i}\right)\right] P_{12} + \sum_{i=1}^{2}\sum_{\substack{j=1\\ j\neq i}}^{2} \frac{\partial f(\mathbf{X}_i)}{\partial \mathbf{v}_i} \cdot \int d^6X_3 \mathbf{a}_{i3} P_{j3} = -\sum_{i=1}^{2}\sum_{\substack{j=1\\ j\neq i}}^{2} \mathbf{a}_{ij} \cdot \frac{\partial}{\partial \mathbf{v}_i} f(\mathbf{X}_i) f(\mathbf{X}_j), \tag{7}$$

in which we have used the notation $P_{ij}=P(\mathbf{X}_i,\mathbf{X}_j)$ and have identified

$$\mathbf{a}_i(\mathbf{x}_i,t) \equiv \int d^6X_j \mathbf{a}_{ij} f(\mathbf{X}_j,t), \tag{8}$$

which is an average of the electrostatic fields surrounding individual particles. Also, since $N \gg \alpha$ we have used $(N-\alpha)/N \approx 1$.

We assume that acceleration due to ensemble averaged forces [i.e., Eq. (8)], which are from potential variations over macroscopic spatial scales, are small. Thus the $\mathbf{a}_i \cdot \partial/\partial \mathbf{v}_i$ terms in Eqs. (6) and (7) can be neglected. Also, the $\mathbf{a}_{ij} \cdot \partial/\partial \mathbf{v}_i$ terms in Eq. (7) can be neglected because they are Λ^{-1} smaller than the $\partial/\partial t + \mathbf{v}_i \cdot \partial/\partial \mathbf{x}_i$ terms. This scaling can be obtained by putting $\Delta x \sim \lambda_D$ into Eq. (2), which gives

$$a_{ij}\frac{\partial/\partial \mathbf{v}_i}{\partial/\partial t} \sim \frac{e^2}{m\lambda_D^2}\frac{1/v_T}{\omega_p} \sim \Lambda^{-1}. \tag{9}$$

Since we only consider electrostatic interactions between particles, $P_{ij}(\mathbf{x}_i,\mathbf{x}_j)=P_{ij}(\mathbf{x}_i-\mathbf{x}_j)$. Furthermore, we apply the Bogoliubov hypothesis: the characteristic time and spatial scales for relaxation of the pair correlation P are much shorter than that for f.[2] We denote the longer time and spatial scales $(\bar{\mathbf{x}},\bar{t})$ and Fourier transform $(\mathcal{F})$ with respect to the shorter spatial scales on which f is approximately constant. We use the definition $\mathcal{F}\{g(\mathbf{x})\}=\hat{g}(\mathbf{k})=\int d^3x \exp(-i\mathbf{k}\cdot\mathbf{x})g(\mathbf{x})$ with inverse $g(\mathbf{x})=(2\pi)^{-3}\int d^3k \exp(i\mathbf{k}\cdot\mathbf{x})\hat{g}(\mathbf{k})$. The double Fourier transform is then

$$\mathcal{F}_{12}\{h(\mathbf{x}_1,\mathbf{x}_2)\} = \hat{h}(\mathbf{k}_1,\mathbf{k}_2) = \int d^3x_1 d^3x_2 e^{-i(\mathbf{k}_1\cdot\mathbf{x}_1+\mathbf{k}_2\cdot\mathbf{x}_2)} h(\mathbf{x}_1,\mathbf{x}_2). \tag{10}$$

Applying these approximations and using the identities $\mathcal{F}_{12}\{h(\mathbf{x}_1-\mathbf{x}_2)\}=(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2)\hat{h}(\mathbf{k}_1)$, $\int d^3x h_1(\mathbf{x})h_2(\mathbf{x}) =(2\pi)^{-3}\int d^3k_1 \hat{h}_1(-\mathbf{k}_1)\hat{h}_2(\mathbf{k}_1)$, and $\mathcal{F}_{12}\{\int d^3x_3 h_1(\mathbf{x}_1-\mathbf{x}_3)h_2(\mathbf{x}_2-\mathbf{x}_3)\}=(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2)\hat{h}_1(\mathbf{k}_1)\hat{h}_2(-\mathbf{k}_1)$, for any functions h_1 and h_2, Eqs. (6) and (7) can be written

$$\left(\frac{\partial}{\partial \bar{t}} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \bar{\mathbf{x}}_1}\right) f(\bar{\mathbf{x}}_1,\mathbf{v}_1,\bar{t}) = C(f_1) = -\frac{\partial}{\partial \mathbf{v}_1}\cdot \mathbf{J}_v \tag{11}$$

and

$$\left[\frac{\partial}{\partial t} + L_1(\mathbf{k}_1) + L_2(-\mathbf{k}_1)\right]\hat{P}_{12}(\mathbf{k}_1,\mathbf{v}_1,\mathbf{v}_2,t) = \hat{S}(\mathbf{k}_1,\mathbf{v}_1,\mathbf{v}_2,\bar{t}). \tag{12}$$

Here $\mathbf{J}_v$ is the collisional current

$$\mathbf{J}_v \equiv \frac{4\pi q_1 q_2}{m_1}\int \frac{d^3k_1}{(2\pi)^3}\frac{-i\mathbf{k}_1}{k_1^2}\int d^3v_2 \hat{P}_{12}(\mathbf{k}_1,\mathbf{v}_1,\mathbf{v}_2,t), \tag{13}$$

L_j is the integral operator

$$L_j(\mathbf{k}_1) \equiv i\mathbf{k}_1\cdot\mathbf{v}_j - i\frac{4\pi q_1 q_2}{m_j}\frac{\mathbf{k}_1}{k_1^2}\cdot\frac{\partial f(\mathbf{v}_j)}{\partial \mathbf{v}_j}\int d^3v_j \tag{14}$$

and $\hat{S}$ is the source term for the pair correlation function equation

$$\hat{S}(\mathbf{k}_1,\mathbf{v}_1,\mathbf{v}_2) = 4\pi i q_1 q_2 \frac{\mathbf{k}_1}{k_1^2}\cdot\left(\frac{1}{m_1}\frac{\partial}{\partial \mathbf{v}_1} - \frac{1}{m_2}\frac{\partial}{\partial \mathbf{v}_2}\right) f(\mathbf{v}_1)f(\mathbf{v}_2). \tag{15}$$

We next use Eqs. (12) and (13) to solve for the collision operator, which is the right side of Eq. (11).

After Laplace transforming with respect to the fast time scale t, Eq. (12) can be written formally as

$$\hat{P}_{12}(\mathbf{k}_1,\omega) = \frac{\tilde{P}_{12}(\mathbf{k}_1,t=0) - \hat{S}/i\omega}{-i\omega + L_1(\mathbf{k}_1) + L_2(-\mathbf{k}_1)}, \tag{16}$$

in which the velocity dependence of $\hat{P}_{12}$, $\hat{S}$, and L has been suppressed for notational convenience. In the following, we neglect the initial pair correlation term $\tilde{P}_{12}(t=0)$ because it is smaller in plasma parameter than the continually evolving collisional source term $\hat{S}$. In Davidson's approach to quasilinear theory, which is a collisionless description of wave-particle interactions, the collisional source term is neglected.[22] Keeping the initial pair correlation term leads to a diffusion equation.[22] Here we are interested in a collision operator.

The $1/[-i\omega+L_1(\mathbf{k}_1)+L_2(-\mathbf{k}_1)]$ part of Eq. (16) is an operator that acts on $-\hat{S}/i\omega$. It can be written[6]

$$\frac{1}{-i\omega+L_1(\mathbf{k}_1)+L_2(-\mathbf{k}_1)} = \frac{1}{(2\pi)^2}\int_{\mathcal{C}_1}\int_{\mathcal{C}_2}\frac{d\omega_1 d\omega_2}{-i(\omega-\omega_1-\omega_2)} \cdot \frac{1}{[-i\omega_1+L_1(\mathbf{k}_1)]}\frac{1}{[-i\omega_2+L_2(-\mathbf{k}_1)]}, \tag{17}$$

in which the contours $\mathcal{C}_1$ and $\mathcal{C}_2$ must be chosen such that $\Im\{\omega\}>\Im\{\omega_1+\omega_2\}$. Frieman and Rutherford[6] showed that

$$\frac{1}{-i\omega_1+L_1(\mathbf{k}_1)} = \frac{i}{\omega_1-\mathbf{k}_1\cdot\mathbf{v}_1}\left\{1-\frac{4\pi q_1 q_2}{m_1 k_1^2}\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_1)/\partial\mathbf{v}_1}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)} \times\int\frac{d^3v_1}{\omega_1-\mathbf{k}_1\cdot\mathbf{v}_1}\right\}, \tag{18}$$

in which

$$\hat{\varepsilon}(\mathbf{k}_1,\omega_1) \equiv 1+\frac{4\pi q_1 q_2}{m_1 k_1^2}\int d^3v_1\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_1)/\partial\mathbf{v}_1}{\omega_1-\mathbf{k}_1\cdot\mathbf{v}_1}. \tag{19}$$

The equivalent expressions for $1/[-i\omega_2+L_2(-\mathbf{k}_1)]$ and $\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)$ are obtained by the substitutions $\mathbf{v}_1\leftrightarrow\mathbf{v}_2$, $\omega_1\leftrightarrow\omega_2$, $m_1\leftrightarrow m_2$, and $\mathbf{k}_1\leftrightarrow-\mathbf{k}_1$.

We call $\mathcal{R}\equiv 1/[-i\omega_1+L_1(\mathbf{k}_1)][-i\omega_2+L_2(-\mathbf{k}_1)]$, the Frieman–Rutherford operator[6] and require $\mathcal{R}\{\hat{S}\}$. Using Eq. (18), the equivalent form for the $1/[-i\omega_2+L_2(-\mathbf{k}_1)]$ term and the source term of Eq. (15), produces an expression for $\mathcal{R}\{\hat{S}\}$ with eight terms. These can be expanded to an expression with twelve terms by identifying four terms with parts that can be written $\hat{\varepsilon}-1$. These are identified using Eq. (19). Eight of the remaining terms cancel and the surviving terms can be written

$$\mathcal{R}\{\hat{S}\} = \frac{-4\pi i q_1 q_2/k_1^2}{(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_1)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)} \times\left\{\left[\frac{\mathbf{k}_1\cdot\partial/\partial\mathbf{v}_1}{m_1\hat{\varepsilon}(\mathbf{k}_1,\omega_1)}-\frac{\mathbf{k}_1\cdot\partial/\partial\mathbf{v}_2}{m_2\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)}\right]f(\mathbf{v}_1)f(\mathbf{v}_2) + \frac{4\pi q_1 q_2}{m_1 m_2 k_1^2}\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_1)/\partial\mathbf{v}_1}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)}\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_2)/\partial\mathbf{v}_2}{\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)} \times\int d^3v_2\frac{f(\mathbf{v}_2)(\omega_1+\omega_2)}{(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_2)}\right\}. \tag{20}$$

For $\mathbf{J}_v$ in Eq. (13), we need $\int d^3v_2\hat{P}_{12}(\mathbf{k}_1,t)$ which is

$$\int d^3v_2\hat{P}_{12}(\mathbf{k},t) = \int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\int d^3v_2\mathcal{R}\{\hat{S}\} \times\int\frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega(\omega-\omega_1-\omega_2)} = \int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\int d^3v_2 \times\frac{(-i)\mathcal{R}\{\hat{S}\}}{\omega_1+\omega_2}[1-e^{-i(\omega_1+\omega_2)t}]. \tag{21}$$

The $\int d^3v_2\mathcal{R}\{\hat{S}\}$ term in Eq. (21) can be simplified using Eqs. (19) and (20), which yields

$$\int d^3v_2\mathcal{R}\{\hat{S}\} = \frac{4\pi i q_1 q_2}{k_1^2}\int d^3v_2\mathbf{k}_1\cdot\left(\frac{1}{m_1}\frac{\partial}{\partial\mathbf{v}_1}-\frac{1}{m_2}\frac{\partial}{\partial\mathbf{v}_2}\right)f(\mathbf{v}_1)f(\mathbf{v}_2) \cdot\frac{\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)-(\omega_1+\omega_2)}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_1)(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_2)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)}. \tag{22}$$

Putting Eq. (22) into Eq. (21), the terms with $\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)$ in the numerator vanish upon completing the ω_2 integral. Inserting the remaining terms into Eq. (13), we find that the collisional current can be written in the Landau form

$$\mathbf{J}_v = \int d^3v_2\mathcal{Q}(\mathbf{v}_1,\mathbf{v}_2)\cdot\left(\frac{1}{m_2}\frac{\partial}{\partial\mathbf{v}_2}-\frac{1}{m_1}\frac{\partial}{\partial\mathbf{v}_1}\right)f(\mathbf{v}_1)f(\mathbf{v}_2), \tag{23}$$

which has both a diffusion component (due to the $\partial/\partial\mathbf{v}_1$ term) and a drag component (due to the $\partial/\partial\mathbf{v}_2$ term). Here $\mathcal{Q}$ is the tensor kernel

$$\mathcal{Q}(\mathbf{v}_1,\mathbf{v}_2) = \frac{(4\pi)^2 q_1^2 q_2^2}{m_1}\int\frac{d^3k_1}{(2\pi)^3}\frac{-i\mathbf{k}_1\mathbf{k}_1}{k_1^4}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi} \cdot\frac{(1-e^{-i(\omega_1+\omega_2)t})(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_1)}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_1)(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_2)\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_1)}. \tag{24}$$

Of the four terms in the numerator of Eq. (24), the two proportional to $\mathbf{k}_1\cdot\mathbf{v}_1$ have overall odd parity in $\mathbf{k}_1$ and vanish upon doing the $\mathbf{k}_1$ integral. The term with just ω_2 vanishes for stable plasmas and is much smaller than the exponentially growing terms for unstable plasmas. Thus it can be neglected. The collisional kernel can then be written in the form

$$\mathcal{Q}(\mathbf{v}_1,\mathbf{v}_2) = \frac{(4\pi)^2 q_1^2 q_2^2}{m_1}\int \frac{d^3k_1}{(2\pi)^3}\frac{-i\mathbf{k}_1\mathbf{k}_1}{k_1^4}p_1(\mathbf{k}_1)p_2(\mathbf{k}_1), \tag{25}$$

in which p_1 and p_2 are defined by

$$p_1(\mathbf{k}_1) = \int \frac{d\omega_1}{2\pi}\frac{e^{-i\omega_1 t}}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_1)(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_2)} \tag{26}$$

and

$$p_2(\mathbf{k}_1) = \int \frac{d\omega_2}{2\pi}\frac{\omega_2 e^{-i\omega_2 t}}{\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_1)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)}. \tag{27}$$

Equations (25)–(27) are identical to Eqs. (25)–(27) of Ref. 1, which were obtained using a test-particle method. The inverse Laplace transforms are carried out accounting for the poles at $\omega = \pm\mathbf{k}\cdot\mathbf{v}$, which leads to the conventional Lenard–Balescu collisional kernel, and for poles at $\hat{\varepsilon}=0$. If instabilities are present, the poles at $\hat{\varepsilon}=0$ produce temporally growing responses.[1]

We make a final substitution in which we identify the species that we have labeled $f(\mathbf{v}_1)$ as species s. The species that interacts with s, which has been labeled $f(\mathbf{v}_2)$ up to now, we label s'. The species s' represent the entire plasma (including s itself) and can be split into different components (i.e., individual s'). Thus, the total s response is due to the sum of the s' components. We also drop the subscripts on $\mathbf{k}_1$ and $\mathbf{v}_1$ and label $\mathbf{v}_2$ as $\mathbf{v}'$.

After these substitutions, the final kinetic equation for species s is $\partial f_s/\partial t + \mathbf{v}\cdot\partial f_s/\partial\mathbf{x} = C(f_s) = \Sigma_{s'}C(f_s,f_{s'})$ in which

$$C(f_s,f_{s'}) = -\frac{\partial}{\partial\mathbf{v}}\cdot\int d^3v'\,\mathcal{Q}\cdot\left(\frac{1}{m_{s'}}\frac{\partial}{\partial\mathbf{v}'} - \frac{1}{m_s}\frac{\partial}{\partial\mathbf{v}}\right)f_s(\mathbf{v})f_{s'}(\mathbf{v}'), \tag{28}$$

is the component collision operator describing collisions between species s and s' and $\mathcal{Q}=\mathcal{Q}_{\text{LB}}+\mathcal{Q}_{\text{IE}}$ is the collisional kernel. The collisional kernel consists of the Lenard–Balescu term

$$\mathcal{Q}_{\text{LB}} = \frac{2q_s^2 q_{s'}^2}{m_s}\int d^3k\frac{\mathbf{k}\mathbf{k}}{k^4}\frac{\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]}{|\hat{\varepsilon}(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2}, \tag{29}$$

that describes the conventional Coulomb scattering of individual particles and the instability-enhanced term

$$\mathcal{Q}_{\text{IE}} = \frac{2q_s^2 q_{s'}^2}{\pi m_s}\int d^3k\frac{\mathbf{k}\mathbf{k}}{k^4}\sum_j \frac{\gamma_j}{(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v})^2+\gamma_j^2}\times\frac{\exp(2\gamma_j t)}{[(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v}')^2+\gamma_j^2]|\partial\hat{\varepsilon}(\mathbf{k},\omega)/\partial\omega|_{\omega_j}^2}, \tag{30}$$

that describes the scattering of particles by collective fluctuations. We can also write the dielectric function in the familiar form

$$\hat{\varepsilon}(\mathbf{k},\omega) = 1 + \sum_{s'}\frac{4\pi q_{s'}^2}{k^2 m_{s'}}\int d^3v\frac{\mathbf{k}\cdot\partial f_{s'}(\mathbf{v})/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}}. \tag{31}$$

We have used the notation $\omega_j = \omega_{R,j}+i\gamma_j$ where $\omega_{R,j}$ and γ_j are the real and imaginary parts of the jth root of the dielectric function Eq. (31).

The total collision operator $C(f_s)$ for the evolution equation of species s is a sum of the collision operators describing collisions between s and each species s' (including itself $s'=s$); thus $C(f_s)=\Sigma_{s'}C(f_s,f_{s'})$. This total collision operator appears from Eqs. (28)–(30) to have four terms: terms for "drag" and "diffusion" (from the $\partial/\partial\mathbf{v}'$ and $\partial/\partial\mathbf{v}$ derivatives, respectively) using both the Lenard–Balescu collisional kernel of Eq. (29), and the instability-enhanced collisional kernel of Eq. (30). However, there are actually only three nonzero terms because the total instability-enhanced contribution to drag vanishes. To show this, we write the total instability-enhanced collision operator as

$$C_{\text{IE}}(f_s) = -\frac{\partial}{\partial\mathbf{v}}\cdot\sum_{s'}\int d^3v'\,\mathcal{Q}_{\text{IE}}\cdot\left(\frac{1}{m_{s'}}\frac{\partial}{\partial\mathbf{v}'} - \frac{1}{m_s}\frac{\partial}{\partial\mathbf{v}}\right)f_s(\mathbf{v})f_{s'}(\mathbf{v}') \tag{32}$$

$$= \frac{\partial}{\partial\mathbf{v}}\cdot\left[\mathcal{D}_{\text{IE,diff}}\cdot\frac{\partial f_s(\mathbf{v})}{\partial\mathbf{v}}\right] - \frac{\partial}{\partial\mathbf{v}}\cdot[\mathbf{D}_{\text{IE,drag}}f_s(\mathbf{v})], \tag{33}$$

in which

$$\mathcal{D}_{\text{IE,diff}} = \sum_{s'}\int d^3v'\,\mathcal{Q}_{\text{IE}}\frac{f_{s'}(\mathbf{v}')}{m_s} \tag{34}$$

and

$$\mathbf{D}_{\text{IE,drag}} = \sum_{s'}\int d^3v'\,\mathcal{Q}_{\text{IE}}\cdot\frac{1}{m_{s'}}\frac{\partial f_{s'}(\mathbf{v}')}{\partial\mathbf{v}'}. \tag{35}$$

Evaluating the dielectric function, Eq. (31), at its roots (ω_j) and multiplying by $(\omega_j-\mathbf{k}\cdot\mathbf{v})^*/(\omega_j-\mathbf{k}\cdot\mathbf{v})^*$ inside the integral gives

$$\hat{\varepsilon}(\mathbf{k},\omega_j) = 1 + \sum_{s'}\frac{4\pi q_{s'}^2}{k^2 m_{s'}}\int d^3v'\times\frac{(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v}'-i\gamma_j)\mathbf{k}\cdot\partial f_{s'}/\partial\mathbf{v}'}{(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v}')^2+\gamma_j^2}. \tag{36}$$

The real and imaginary parts of Eq. (36) individually vanish; $\Re\{\hat{\varepsilon}(\mathbf{k},\omega_j)\} = \Im\{\hat{\varepsilon}(\mathbf{k},\omega_j)\}=0$. The imaginary part is

$$\Im\{\hat{\varepsilon}(\mathbf{k},\omega_j)\} = \sum_{s'}\frac{4\pi q_{s'}^2}{k^2 m_{s'}}\int d^3v'\frac{(-\gamma_j)\mathbf{k}\cdot\partial f_{s'}(\mathbf{v}')/\partial\mathbf{v}'}{(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v}')^2+\gamma_j^2}. \tag{37}$$

Equation (37) shows that the term proportional to $\omega_{R,j}$ in Eq. (36) is zero; thus the real part of Eq. (36) can be written

$$\Re\{\hat{\varepsilon}(\mathbf{k},\omega_j)\} = 1 - \sum_{s'} \frac{4\pi q_{s'}^2}{k^2 m_{s'}} \int d^3v' \frac{\mathbf{k}\cdot\mathbf{v}'\mathbf{k}\cdot\partial f_{s'}(\mathbf{v}')/\partial\mathbf{v}'}{(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v}')^2+\gamma_j^2}. \tag{38}$$

Putting Eq. (30) into $\mathbf{D}_{\mathrm{IE,drag}}$ in Eq. (35), we find $\mathbf{D}_{\mathrm{IE,drag}}(f_s) \propto \Im\{\hat{\varepsilon}(\mathbf{k},\omega_j)\}=0$. Thus, only a diffusion term survives in the instability-enhanced portion of the total collision operator $C_{\mathrm{IE}}(f_s)$.

Although the instability-enhanced drag is zero in the total collision operator $C(f_s)$, it is not necessarily zero in each component collision operator $C(f_s,f_{s'})$. Identifying the component collision operators in the form of Eq. (28) is particularly useful because they do not each cause evolution of f_s on similar time scales. For example, like-particle collisions $(s=s')$ tend to dominate unlike-particle collisions for short times. The time scale for which unlike-particle collisions matter may be longer than those of interest, or those of mechanisms external to the plasma theory such as neutral collisions or losses to boundaries. Thus, although the instability-enhanced contribution to drag in the total collision operator vanishes, it remains a useful term in describing the individual collision operator components. Both examples discussed in Sec. V are concerned with component collision operators.

As discussed in Ref. 1, time in the $2\gamma_j t$ term of Eq. (30) must be computed in the reference frame of the unstable wave. For convective modes with group velocity $\mathbf{v}_g=\partial\omega_R/\partial\mathbf{k}$, this is

$$2\gamma t = 2\int_{\mathbf{x}_o(\mathbf{k})}^{\mathbf{x}} d\mathbf{x}'\cdot\frac{\mathbf{v}_g\gamma}{|\mathbf{v}_g|^2}, \tag{39}$$

in which $\mathbf{x}_o(\mathbf{k})$ is the location where the mode with wavenumber $\mathbf{k}$ becomes unstable, $\mathbf{x}$ is the spatial variable and $\mathbf{x}'$ is the path that the wave travels from $\mathbf{x}_o$ to $\mathbf{x}$. Thus, for convective instabilities f does not change in time at a fixed spatial location, but does change as a function of position along the convection path of the unstable fluctuations. For inhomogeneous media, different wavenumbers $\mathbf{k}$ may become unstable at different spatial locations, thus $\mathbf{x}_o(\mathbf{k})$.

III. RELATING $C_{\mathrm{IE}}(f_s)$ AND QUASILINEAR THEORY

The instability-enhanced term of the total collision operator, Eq. (33), is a diffusion equation similar to quasilinear theory.[8,9] In fact, $C_{\mathrm{IE}}(f_s)$ may be interpreted as a quasilinear operator in which the continuing source of fluctuations is self-consistently calculated from discrete particle motion in the plasma. Conventional quasilinear theory does not identify an origin of fluctuations and this must be supplied external to the theory. In this section, we show that if the source of fluctuations is internal to the plasma, rather than from externally applied sources, the instability-enhanced term of the kinetic theory of Sec. II fits into the conventional quasilinear formalism, but with a self-consistent determination of the spectral energy density.

Conventional quasilinear theory is based on the Vlasov equation $\partial f_s/\partial t+\mathbf{v}\cdot\partial f_s/\partial\mathbf{x}+(q_s/m_s)\mathbf{E}\cdot\partial f_s/\partial\mathbf{v}=0$. By separating the distribution function and electric field into smoothed and fluctuating components $f_s=f_{s,o}+f_{s,1}$, such that $f_{s,o}=\langle f_s\rangle$ where the $\langle\cdots\rangle$ represents a spatial average, and applying the quasilinear approximation, which assumes $\mathbf{E}_1 f_{s,1}-\langle\mathbf{E}_1 f_{s,1}\rangle \ll \mathbf{E}_1 f_{s,o}$, one can derive the quasilinear diffusion equation[8,9] of species s

$$\frac{\partial f_{s,o}}{\partial t} = \frac{\partial}{\partial\mathbf{v}}\cdot\mathcal{D}_v\cdot\frac{\partial f_{s,o}}{\partial\mathbf{v}}. \tag{40}$$

The velocity-space diffusion coefficient is

$$\mathcal{D}_v = \frac{q_s^2}{m_s^2}8\pi\sum_j\int d^3k\frac{\mathbf{kk}}{k^2}\frac{\gamma_j\mathcal{E}_j(\mathbf{k})}{[(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v})^2+\gamma_j^2]}, \tag{41}$$

in which j represents the unstable modes and the spectral energy density is defined as

$$\mathcal{E}_j^{\mathrm{ql}}(\mathbf{k}) = \frac{|E_1(t=0)|^2}{(2\pi)^3V}\frac{e^{2\gamma_j t}}{8\pi}. \tag{42}$$

The term $|E_o|^2$ is the initial electrostatic fluctuation level that must be supplied external to this theory.

An alternative approach to quasilinear theory that uses the BBGKY hierarchy has been provided by Davidson.[22] This can be derived by the same methods as in Sec. II, except taking only the initial pair correlation term $\tilde{P}(t=0)$ in Eq. (16) instead of the continuous source term $\hat{S}$ that was used in Sec. II, and keeping only terms that grow as $\exp(2\gamma_j t)$. Removing the source term, which is larger than the initial pair correlation by $\mathcal{O}(\Lambda)$, effectively removes collisions and leaves a Vlasov model. The resulting diffusion equation has the same form as Eqs. (40) and (41), but with the spectral energy density redefined as a function of the initial pair correlation instead of the initial fluctuation level[22]

$$\mathcal{E}_j^{\mathrm{dv}}(\mathbf{k}) = \sum_{s'}\frac{q_{s'}^2}{4\pi^2k^2|\partial\hat{\varepsilon}/\partial\omega|_{\omega_j}^2}\int d^3v \times\int d^3v'\frac{\tilde{P}(\mathbf{k},\mathbf{v},\mathbf{v}',t=0)e^{2\gamma_j t}}{(\omega_j-\mathbf{k}\cdot\mathbf{v})(\omega_j^*-\mathbf{k}\cdot\mathbf{v}')}. \tag{43}$$

The instability-enhanced contribution to the total collision operator, Eq. (33), is a diffusion equation that also fits into the quasilinear formalism of Eqs. (40) and (41), but with the spectral energy density self-consistently calculated to be

$$\mathcal{E}_j^{\mathrm{kin}}(\mathbf{k}) = \sum_{s'}\frac{q_{s'}^2}{4\pi^2k^2|\partial\hat{\varepsilon}/\partial\omega|_{\omega_j}^2}\int d^3v'\frac{f_{s'}(\mathbf{v}')e^{2\gamma_j t}}{(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v}')^2+\gamma_j^2}. \tag{44}$$

An important feature of Eq. (44) is that it does not depend on specifying an initial electrostatic fluctuation level, as Eq. (42) requires, or an initial pair correlation function, as Eq. (43) requires. Equation (44) also shows that when fluctuations originate from discrete particle motion, the spectral energy density has a particular dependence on $\mathbf{k}$ that is determined by the plasma dielectric function. This $\mathbf{k}$ dependence cannot be captured by the conventional quasilinear theory, Eq. (42), which typically proceeds by specifying a constant for $|E_1(t=0)|^2$ to determine the spectral energy density.

IV. PHYSICAL PROPERTIES OF $C(f)$

The kinetic equation derived in Sec. II obeys certain physical properties such as conservation laws and the Boltzmann $\mathcal{H}$-theorem. A brief overview of these properties is provided in this section along with a discussion of how the plasma evolves to equilibrium and how these properties relate to those of the effective collision operator in conventional quasilinear theory.

(a) *Density is conserved.* Collisions do not create or destroy particles or cause them to change species. For collisions between species s and s', this can be expressed mathematically as

$$\int d^3v C(f_s,f_{s'}) = 0, \tag{45}$$

which also implies the less restrictive conditions that the species s density is conserved $\int d^3v df_s/dt = \int d^3v \Sigma_{s'} C(f_s,f_{s'})=0$ and the total plasma density is conserved $\int d^3v \Sigma_s df_s/dt=0$. This is true of both the $C_{\rm LB}$ and $C_{\rm IE}$ terms individually. The conventional quasilinear theory summarized in Sec. III, does not distinguish the species s', so one cannot show Eq. (45), but it does satisfy that the species s and total plasma density are conserved.

Proof: Equation (45) follows directly from writing $C(f_s,f_{s'})$ in the form of a divergence of the collisional current $C(f_s,f_{s'})=-\nabla_v\cdot\mathbf{J}_v^{s/s'}$. The integral over velocity vanishes due to the divergence theorem since $\mathbf{J}_v^{s/s'}$ is zero at infinity.

(b) *Momentum is conserved.* Momentum lost from species s due to collisions of species s with species s' is gained by species s'. Mathematically this is expressed as

$$\int d^3v m_s\mathbf{v}C(f_s,f_{s'}) + \int d^3v m_{s'}\mathbf{v}C(f_{s'},f_s) = 0. \tag{46}$$

Equation (46) implies that the total momentum is conserved: $\int d^3v \Sigma_s m_s df_s/dt=0$. Only total momentum can be shown to be conserved within the conventional quasilinear theory.

Proof: Equation (46) follows from first integrating by parts to show $\int d^3v m_s\mathbf{v}C(f_s,f_{s'})=m_s\int d^3v \mathbf{J}_v^{s/s'}$, which is

$$\int d^3v m_s\mathbf{v}C(f_s,f_{s'}) = -\int d^3v\int d^3v' m_s \mathcal{Q}_{s,s'}\cdot\mathcal{X}_{s,s'}, \tag{47}$$

where we have defined

$$\mathcal{X}_{s,s'}(\mathbf{v},\mathbf{v}') = \frac{f_{s'}(\mathbf{v}')}{m_s}\frac{\partial f_s(\mathbf{v})}{\partial \mathbf{v}} - \frac{f_s(\mathbf{v})}{m_{s'}}\frac{\partial f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'}. \tag{48}$$

An expression for $\int d^3v m_{s'}\mathbf{v}C(f_{s'},f_s)$ is obtained by the substitutions $s\leftrightarrow s'$ and $\mathbf{v}\leftrightarrow\mathbf{v}'$ in Eq. (47). Using the properties $m_{s'}\mathcal{Q}_{s',s}=m_s\mathcal{Q}_{s,s'}$ and $\mathcal{X}_{s,s'}(\mathbf{v},\mathbf{v}')=-\mathcal{X}_{s',s}(\mathbf{v}',\mathbf{v})$ in the result and adding it to Eq. (47) yields the conservation of momentum expression of Eq. (46).

(c) *The sum of particle and wave energy is conserved.* The energy lost by species s due to conventional Coulomb collisions of s with s' (described by the Lenard–Balescu operator) is gained by s'. Mathematically this can be written

$$\int d^3v \frac{1}{2}m_s v^2 C_{\rm LB}(f_s,f_{s'}) + \int d^3v\frac{1}{2}m_{s'}v^2 C_{\rm LB}(f_{s'},f_s) = 0, \tag{49}$$

and it implies that $\int d^3v \Sigma_s m_s v^2 C_{\rm LB}(f_s)/2=0$. The instability-enhanced portion of the collision operator shows that a change in total energy in the plasma is balanced by a change in wave energy. Thus we find that the total energy conservation relation is given by

$$\int d^3v \sum_s \frac{1}{2}m_s v^2 C(f_s) = -\frac{\partial}{\partial t}\int d^3k\frac{\mathcal{E}(\mathbf{k})}{k^2}, \tag{50}$$

in which the spectral energy density is defined in terms of Eq. (44). Equation (50) is also satisfied in conventional quasilinear theory.

Proof: Conservation of energy from the Lenard–Balescu collision operator, Eq. (49), follows from first integrating by parts to show $\int d^3v m_s v^2 C(f_s,f_{s'})/2=\int d^3v m_s\mathbf{v}\cdot\mathbf{J}_v^{s/s'}$. Putting in $\mathbf{J}_{\rm LB}^{s/s'}$ and using the same method that was used in the proof of momentum conservation for obtaining an expression for $\int d^3v m_{s'}v^2 C_{\rm LB}(f_{s'},f_s)/2$ yields

$$\int d^3v\frac{v^2}{2}[m_s C_{\rm LB}(f_s,f_{s'}) + m_{s'}C_{\rm LB}(f_{s'},f_s)]$$
$$= -\int d^3v\int d^3v' m_s\mathcal{Q}_{\rm LB}\cdot(\mathbf{v}-\mathbf{v}')\cdot\mathcal{X}_{s,s'}. \tag{51}$$

Since $\mathcal{Q}_{\rm LB}\cdot(\mathbf{v}-\mathbf{v}')=0$, the right side of Eq. (51) vanishes; thus proving Eq. (49) and by a trivial extension $\int d^3v\Sigma_s m_s v^2 C_{\rm LB}(f_s)/2=0$.

The only nonvanishing component of the conservation of energy relation is the instability-enhanced portion which can be written $\int d^3v \Sigma_s m_s v^2 C_{\rm IE}(f_s)/2 = -\int d^3v \Sigma_s m_s \mathbf{v}\cdot\mathcal{D}_{\rm IE,diff}\cdot\partial f_s/\partial\mathbf{v}$. Inserting $\mathcal{D}_{\rm IE,diff}$ from Eq. (34), identifying Eqs. (41) and (44), gives

$$\int d^3v\sum_s\frac{1}{2}m_s v^2 C_{\rm IE}(f_s)$$
$$= -\int d^3k\frac{2\gamma\mathcal{E}(\mathbf{k})}{k^2}\left[\sum_s\frac{4\pi q_s^2}{k^2 m_s}\int d^3v\frac{\mathbf{k}\cdot\mathbf{v}\mathbf{k}\cdot\partial f_s/\partial\mathbf{v}}{(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v})^2+\gamma_j^2}\right]. \tag{52}$$

Equation (38), describing $\Re\{\hat{\varepsilon}(\mathbf{k},\omega_j)\}=0$, shows that the term in square brackets in Eq. (52) is equal to 1. Identifying $2\gamma_j\mathcal{E}=\partial\mathcal{E}/\partial t$ in Eq. (52) from Eq. (44) and adding the condition $\int d^3v\Sigma_s m_s v^2 C_{\rm LB}(f_s)/2=0$ to the result completes the proof of Eq. (50).

(d) *If $f_s\geq 0$ initially, $f_s\geq 0$ for all time.* Proof of this property for the kinetic equation of Sec. II is analogous to that provided by Lenard for the Lenard–Balescu equation.[12] The proof requires that $C(f_s)$ can be written in the Landau form of Eq. (28) and that $\mathcal{Q}$ is a positive definite tensor. These conditions are both satisfied by the generalized collision operator with $\mathcal{Q}=\mathcal{Q}_{\rm LB}+\mathcal{Q}_{\rm IE}$ derived in Sec. II. The expression for $\mathcal{Q}_{\rm IE}$ is only valid for $\gamma_j\geq 0$, as is quasilinear theory. However, one criticism of quasilinear theory is that it does not transition to stable plasmas.[24] If $\gamma_j<0$, the diffusion coefficient is negative and an unphysical equation results. This would be a misapplication, but also illustrates the limits of quasilinear theory because a Coulomb collision operator [i.e., $C_{\mathrm{LB}}(f_s)$] is required near marginal stability.

(e) *The Boltzmann $\mathcal{H}$-theorem is satisfied*. The $\mathcal{H}$-functional for each species s is defined as $\mathcal{H}_s \equiv \int d^3v f_s(\mathbf{v}) \ln f_s(\mathbf{v})$ and the total $\mathcal{H}$ is the sum of the component species $\mathcal{H} = \Sigma_s \mathcal{H}_s$. The Boltzmann $\mathcal{H}$-theorem states that the total $\mathcal{H}$ satisfies $d\mathcal{H}/dt \leq 0$. It is equivalent to stating that entropy always increases until equilibrium in reached.

Proof: The time derivative of $\mathcal{H}_s$ is $d\mathcal{H}_s/dt = \int d^3v [1 + \ln f_s(\mathbf{v})] df_s(\mathbf{v})/dt$. Using the conservation of density property from (a) gives $d\mathcal{H}_s/dt = -\int d^3v \Sigma_{s'} \ln(f_s) \nabla_{\mathbf{v}} \cdot \mathbf{J}_v^{s/s'}$. Integrating by parts yields $d\mathcal{H}_s/dt = \Sigma_{s'} \int d^3v \mathbf{J}_v^{s/s'} \cdot \partial \ln f_s / \partial \mathbf{v}$. We then identify the components of $\mathcal{H}_s$ such that $\mathcal{H}_s = \Sigma_{s'} \mathcal{H}_{s,s'}$. Putting in the kinetic equation of Sec. II gives

$$\frac{d\mathcal{H}_{s,s'}}{dt} = -\int d^3v \int d^3v' \frac{1}{m_s} \frac{\partial \ln f_s(\mathbf{v})}{\partial \mathbf{v}} \cdot (m_s \mathcal{Q}_{s,s'}) \cdot \mathcal{X}_{s,s'}. \tag{53}$$

By interchanging the species $s \leftrightarrow s'$ and dummy integration variables $\mathbf{v} \leftrightarrow \mathbf{v}'$ an expression for $\mathcal{H}_{s',s}$ is obtained

$$\frac{d\mathcal{H}_{s',s}}{dt} = \int d^3v \int d^3v' \frac{1}{m_{s'}} \frac{\partial \ln f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'} \cdot (m_{s'} \mathcal{Q}_{s',s}) \cdot \mathcal{X}_{s,s'}. \tag{54}$$

Using $m_s \mathcal{Q}_{s,s'} = m_{s'} \mathcal{Q}_{s',s}$ in Eq. (54) along with Eq. (53) in $2\mathcal{H} = \Sigma_s \Sigma_{s'} (\mathcal{H}_{s,s'} + \mathcal{H}_{s',s})$ yields

$$2\frac{d\mathcal{H}}{dt} = -\sum_s \sum_{s'} \int d^3v \int d^3v' \frac{\mathcal{X}_{s,s'} \cdot (m_s \mathcal{Q}_{s,s'}) \cdot \mathcal{X}_{s,s'}}{f_s(\mathbf{v}) f_{s'}(\mathbf{v}')}. \tag{55}$$

Since the $\mathcal{Q}_{s,s'}$ of Sec. II is positive-semidefinite and $f_s, f_{s'} \geq 0$ from (d), each term on the right side of Eq. (55) is negative-semidefinite. Thus, we find that the Boltzmann $\mathcal{H}$-theorem is satisfied: $d\mathcal{H}/dt \leq 0$.

(f) *The unique equilibrium distribution function is Maxwellian and the approach to equilibrium is hastened by instabilities.* Equilibrium is established when $d\mathcal{H}/dt = 0$. In the analysis below, we first show that C_{LB} implies that the unique equilibrium state of a plasma is a Maxwellian in which each species has the same temperature and flow velocity. Since instabilities cannot be present near this equilibrium, $C_{\mathrm{IE}} = 0$, and instability-enhanced effects are irrelevant. However, bounded plasmas are rarely in true equilibrium. A much more common concern is to determine the time scales for which equilibration between individual species occurs. The fastest time scales are typically for self-equilibration within species. For example, electrons and ions in a plasma may be in equilibrium with themselves, in which case $d\mathcal{H}_{e,e}/dt = 0$ and $d\mathcal{H}_{i,i}/dt = 0$, but not in equilibrium with each other, so $d\mathcal{H}_{i,e}/dt \neq 0$. In this case, electrons and ions will individually have Maxwellian distributions, but their temperature and flow velocities will not necessarily be the same and instabilities may be present. In the second part of the analysis below, we show that instabilities can significantly shorten the time scale for which individual species reach a Maxwellian quasiequilibrium.

Analysis: First, we consider the final equilibrium state of the plasma from the C_{LB} term of the collision operator. Since each term of Eq. (55) is negative-semidefinite, each must vanish independently in order to reach equilibrium at $d\mathcal{H}/dt = 0$. The terms that tend to zero on the fastest time scale are those describing like-particle collisions $s = s'$. Considering these first, $d\mathcal{H}_{s,s}/dt = 0$ implies that $\mathcal{X}_{s,s} \propto \mathbf{v} - \mathbf{v}'$ [because $\mathcal{Q}_{\mathrm{LB}} \cdot (\mathbf{v} - \mathbf{v}') = 0$]. This can happen only when $f_s(\mathbf{v})$ has the general Maxwellian form $f_{Ms}(\mathbf{v}) = \exp(-A_s v^2/2 + \mathbf{B}_s \cdot \mathbf{v} + C_s)$. Proof of this step is provided by Lenard.[12] Applying the conventional definitions for density $n_s = \int d^3v f_s$, flow velocity $\mathbf{V}_s = \int d^3v \mathbf{v} f_s / n_s$ and thermal speed $v_{Ts}^2 = \frac{2}{3} \int d^3v (\mathbf{v} - \mathbf{V}_s)^2 f_s / n_s = 2T_s/m_s$, the Maxwellian for species s can be written in the familiar form

$$f_{Ms} = \frac{n_s}{\pi^{3/2} v_{Ts}^3} \exp\left[-\frac{(\mathbf{v} - \mathbf{V}_s)^2}{v_{Ts}^2}\right]. \tag{56}$$

On a longer time scale the unlike-particle terms $(s \neq s')$ of Eq. (55) must also vanish for equilibrium to be reached. This implies $\mathcal{X}_{s,s'} \propto \mathbf{v} - \mathbf{v}'$. Putting the individual species Maxwellians of Eq. (56) into this condition gives

$$\left(\frac{\mathbf{v}}{T_s} - \frac{\mathbf{v}'}{T_{s'}}\right) + \left(\frac{\mathbf{V}_{s'}}{T_{s'}} - \frac{\mathbf{V}_s}{T_s}\right) \propto \mathbf{v} - \mathbf{v}', \tag{57}$$

which is satisfied only if $T_s = T_{s'}$ and $\mathbf{V}_s = \mathbf{V}_{s'}$. Thus, the unique equilibrium state of the plasma is that each species have a Maxwellian distribution of the form of Eq. (56) with the same flow velocity and temperature.

Next, we consider the role of instability-enhanced collisions in the equilibration process. The analysis just considered could be repeated by substituting $\mathcal{Q}_{\mathrm{IE}}$ for $\mathcal{Q}_{\mathrm{LB}}$, except that Eq. (30) is not proportional to $\delta[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')]$ and does not satisfy $\mathcal{Q}_{\mathrm{IE}} \cdot (\mathbf{v} - \mathbf{v}') = 0$. However, $\mathcal{Q}_{\mathrm{IE}}$ has a Lorentzian form in velocity space that is very peaked around $\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}') = 0$ and the dominant term can be written in the delta function form. Substituting $\mathcal{Q}_{\mathrm{IE}}$ into Eq. (55) gives an expression that depends on velocity-space integrals in $\mathbf{v}$ and $\mathbf{v}'$ over Lorentzian distributions. We assume that the instabilities are weakly growing and thus satisfy $\gamma_j \ll \omega_{R,j} - \mathbf{k} \cdot \mathbf{v}$. These velocity-space integrals are of the form $\int dx g(x) \Delta / [(x-a)^2 + \Delta^2]$, where $\Delta \ll a$. Here the appropriate substitutions are $\Delta = \gamma_j$, $a = \omega_{R,j}$ and $x = \mathbf{k} \cdot \mathbf{v}$ (or $x = \mathbf{k} \cdot \mathbf{v}'$). They can be approximated by

$$\int_{-\infty}^{\infty} dx g(x) \frac{\Delta}{(a-x)^2 + \Delta^2} \approx \int_{-n\Delta}^{n\Delta} dy g(y+a) \frac{\Delta}{y^2 + \Delta^2}, \tag{58}$$

where n is a number large enough to span most of the integrand. Expanding $g(x)$ about the peak at $x = a$, $g(x) \approx g(a) + g'(a)y + g''(a)y^2/2 + \cdots$, the lowest order term in Eq. (58) gives $g(a) 2 \arctan(n) \approx \pi g(a)$. The second term is zero. The third term gives $g''(a) \Delta^2 [n - \arctan(n)] \sim \mathcal{O}(\Delta^2/a^2)$. Thus, these integrals satisfy

$$\int dx g(x) \frac{\Delta}{(a-x)^2+\Delta^2} \approx \int dx \pi g(x) \delta(x-a) + \mathcal{O}\left(\frac{\Delta^2}{a^2}\right). \tag{59}$$

Using Eq. (59) and the property $\delta(x-a)\delta(x-b) = \delta(x-a)\delta(a-b)$, we find that within the velocity-space integrals of Eq. (55), the $\mathcal{Q}_{\mathrm{IE}}$ term can be written in the form

$$\mathcal{Q}_{\mathrm{IE}} = \sum_j \frac{2q_s^2 q_{s'}^2}{m_s} \int d^3k \frac{\mathbf{kk}}{k^4} \frac{\pi\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]\delta(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v})e^{2\gamma_j t}}{\gamma_j |\partial\hat{\varepsilon}(\mathbf{k},\omega)/\partial\omega|^2_{\omega_j}} + \mathcal{O}\left(\frac{\gamma_j^2}{\omega_{R,j}^2}\right). \tag{60}$$

Since the lowest order term of Eq. (60) satisfies $\mathcal{Q}\cdot(\mathbf{v}-\mathbf{v}')=0$, one can repeat the analysis used for the Lenard–Balescu term to show that the instability-enhanced term drives the plasma to a quasiequilibrium state that is nearly Maxwellian. The correction terms in Eq. (60) will not obey this property and can be expected to cause some deviation from the Maxwellian. In this sense there is no true equilibrium if instabilities are present. However, if the distribution is non-Maxwellian the lowest order term in Eq. (60) drives the distribution toward a Maxwellian on a faster time scale, by a factor of $\mathcal{O}(\omega_{R,j}^2/\gamma_j^2)$, than the correction terms cause a deviation from Maxwellian. Thus, if instabilities are present from the interaction of two different species, for example flow-driven instabilities, the instabilities can shorten the time scale for which each species will self-equilibrate to a Maxwellian. On a typically much longer time scale the different species will also equilibrate with one another. However, in the mean time a quasiequilibrium can be established where each species is Maxwellian, but the flow speeds and temperatures have not yet equilibrated and instabilities can persist. In Sec. V applications are considered where flowing instabilities cause the self-equilibration time scales to shorten, but for which the time scales for different species to equilibrate remain much longer than the time it takes the plasma to flow out of the system.

It may be worth pointing out that one cannot show from conventional quasilinear theory that instability-enhanced collisions drive the plasma toward a Maxwellian. This can be shown within the kinetic theory because kinetic theory distinguishes the origin of fluctuations and thus determines the spectral energy density $\mathcal{E}(\mathbf{k})$ [see Eq. (44)]. Specification of $\mathcal{E}(\mathbf{k})$ is required in order to show that $\mathcal{Q}_{\mathrm{IE}}$ is very peaked in velocity-space about $\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')$, and this is an important property for showing that Maxwellian is the unique equilibrium (or quasiequilibrium depending on the time scale considered).

V. APPLICATIONS

In this section, we briefly review two applications where Eq. (28) has been successfully applied. Both are concerned with scattering phenomena in the plasma-boundary transition region of low temperature plasmas (with $T_e \gg T_i$). In particular, we will be concerned with the presheath region in which a weak electric field accelerates ions from a fluid moment speed of essentially zero in the bulk plasma to near the ion sound speed at the presheath-sheath boundary. The presheath length (l) is approximately an ion-neutral collision length in these plasmas. This is on the 1–10 cm length scale and is typically 10^2-10^3 times longer than a Debye length which is $\sim 0.1-1$ mm in these plasmas.

A. Langmuir's paradox

Langmuir's paradox[17,18] is a measurement of anomalous electron-electron scattering in a low temperature plasma. In particular, Langmuir measured self-equilibration of electrons to a Maxwellian in a discharge of approximately 3 cm diameter when the electron-electron scattering length was calculated to be 30 cm (assuming a stable plasma). Langmuir expected truncation of the electron distribution at an energy corresponding to the sheath potential energy ($\approx 5T_e$ for mercury) because electrons exceeding this energy escape the sheath and rapidly leave the plasma. The fact that Langmuir measured a Maxwellian for energies beyond the sheath energy suggested that some mechanism for scattering electrons must be present that is at least ten times more frequent than scattering by the Coulomb interaction of individual particles.

Langmuir's mercury discharge had a plasma density of $n \approx 10^{11}$ cm^{-3}, neutral density $\approx 10^{13}$ cm^{-3} (0.3 mTorr) and an electron temperature of $T_e \approx 2$ eV.[17] Although the ion temperature could not be diagnosed at the time of Langmuir's experiment, it has since been shown that in these discharges ions are typically in thermal equilibrium with the neutral particles. Thus the ion temperature is near room temperature and satisfies $T_e \gg T_i$.

Reference 15 applied Eq. (28) to show that ion-acoustic instabilities, which are present in the presheath, could significantly enhance the electron-electron collision frequency in Langmuir's discharge. The collision frequency was shown to be a sum of contributions from the conventional Coulomb scattering of individual particles (the Lenard–Balescu term) and the instability-enhanced interaction from the scattering of particles from the collective wave motion: $\nu^{e/e} = \nu_{\mathrm{LB}}^{e/e} + \nu_{\mathrm{IE}}^{e/e}$. The stable plasma contribution is calculated using Eq. (29), which for thermal particles gives

$$\nu_{\mathrm{LB}}^{e/e} \sim \frac{\omega_{pe}}{8\pi n \lambda_{De}^3} \ln \Lambda. \tag{61}$$

For Langmuir's discharge, the electron-electron collision length is $\lambda_{\mathrm{LB}}^{e/e} \approx v_{Te}/\nu_{\mathrm{LB}}^{e/e} \approx 28$ cm, if the plasma is stable.

The instability-enhanced collision frequency was calculated using the plasma dielectric function with fluidlike ions streaming with a fluid velocity $\mathbf{V}_i$ and nearly adiabatic electrons. Instability is caused by a small nonadiabatic electron response. The dielectric function for ion-acoustic instabilities is given by

$$\hat{\varepsilon} = 1 + \frac{1}{k^2\lambda_{De}^2} - \frac{\omega_{pi}^2}{(\omega - \mathbf{k}\cdot\mathbf{V}_i)^2} + i\frac{\sqrt{\pi}}{k^2\lambda_{De}^2}\frac{\omega}{kv_{Te}}. \tag{62}$$

The roots of this equation, $\hat{\varepsilon}=0$, are given by the dispersion relation

$$\omega_{\pm} = \left(\mathbf{k}\cdot\mathbf{V}_i \pm \frac{kc_s}{\sqrt{1+k^2\lambda_{De}^2}}\right)\left(1 \mp i\frac{\sqrt{\pi m_e/8M_i}}{(1+k^2\lambda_{De}^2)^{3/2}}\right). \quad (63)$$

Ion-acoustic instabilities are present if the ion fluid speed satisfies $|\mathbf{k}\cdot\mathbf{V}_i| > kc_s/\sqrt{1+k^2\lambda_{De}^2}$.

Using Eqs. (62) and (63) in Eq. (30) leads to an expression for the instability-enhanced collision frequency for electrons near thermal speed

$$\nu_{\mathrm{IE}}^{e/e} \sim \frac{\nu_{\mathrm{LB}}^{e/e}}{8\ln\Lambda}\frac{1+2\kappa_c^2}{(1+\kappa_c^2)^2}\exp\left(\eta\frac{z}{l}\right). \quad (64)$$

Here $\eta \equiv l\sqrt{\pi m_e/16M_i}/\lambda_{De}$ where $z=0$ is the location along the presheath that instabilities first become excited and $\kappa_c \equiv \sqrt{c_s^2/V_i^2-1}$ accounts for the k-space cutoff of the instabilities; it is valid for $V_i \le c_s$ (the presheath region), otherwise $\kappa_c=0$. For Langmuir's discharge, $\nu_{\mathrm{IE}}^{e/e}/\nu_{\mathrm{LB}}^{e/e}$ gets as large as 100 near the presheath-sheath boundary and it is at least 10 for much of the presheath.[15] Thus, the instability-enhanced collisions shrink the effective electron-electron collision length to within the dimensions of Langmuir's discharge. Since $\gamma^2/\omega_R^2 \sim m_e/M_i \sim 10^{-4}$, the correction terms of Eq. (60) are very small and the instability-enhanced term drives the electrons to a Maxwellian distribution. Thus, the plasma reaches a quasiequilibrium state where electrons and ions have Maxwellian distributions, but with unequal flow speeds and temperatures.

For this plasma with $n\lambda_{De}^3 \approx 3\times10^3$, the kinetic theory is valid for $\eta z/l \lesssim 55$.[1,15] For longer growth lengths, the fluctuation amplitude is expected to be nonlinear and this kinetic theory is no longer valid. However, in this calculation the growth distance (for convective instabilities) is restricted to the presheath length. After propagating through the presheath, the waves are lost to the plasma boundaries. For this presheath $\eta z/l \lesssim 10$; the kinetic theory is well suited to this problem.[15]

B. Determining the Bohm criterion

The Bohm criterion[25] states that the ion fluid speed must be supersonic at the presheath-sheath boundary: $V \ge \sqrt{T_e/M_i}$ (equality typically holds). It is important because it determines the flux and energy of ions as they leave a plasma. However when the Bohm criterion is generalized to multiple ion species, the result[26]

$$\sum_{i=1}^{N}\frac{n_{io}}{n_{eo}}\frac{c_{s,i}^2}{V_i^2} \le 1, \quad (65)$$

is a single equation in N unknowns, where N is the number of different ion species. Thus Eq. (65) does not uniquely determine the speed of each ion species as it leaves the plasma (even assuming equality holds, as in the single species case). In Eq. (65) we have assumed that $T_e \gg T_i$, which is valid for the plasmas we are concerned with here.

Franklin[27] has studied the plasma-boundary transition theoretically and proposed that each ion species enters the sheath with a fluid speed near its individual sound speed $V_i \approx \sqrt{T_e/M_i}$. In his work, Franklin accounted for ion drag due to ionization sources and ion-neutral collisions, but neglected ion-ion drag. In a stable plasma with typical low-temperature parameters, ion-ion friction is normally at least ten times smaller than other terms in the fluid momentum balance equation; neglecting it is justified. Experimental studies[28] have used laser-induced fluorescence to reveal that the ion flows at the sheath edge can be far from their individual sound speeds. The measured flow speeds in these experiments (which are in a low-temperature regime $T_e \sim 1$ eV, $T_i \sim 0.02$ eV) tend to be closer to a common "system" sound speed $c_s = \sqrt{\sum_i c_{s,i}^2 n_i/n_e}$ than their individual sound speeds $c_{s,i}$. This suggests that some anomalous ion-ion friction may be present.

Reference 16 applied Eq. (28) to calculate the collisional friction force between ion species using parameters of a plasma from the experimental literature:[28] Ar^+ and Xe^+ ions with equal densities, $T_e=0.7$ eV, $T_i=0.02$ eV, and a neutral pressure of 0.7 mTorr. It was shown that ion-ion streaming instabilities can be present in the presheath and that these instabilities lead to a rapid enhancement of the collisional friction between ion species. This collisional friction force $[\mathbf{R}^{s-s'} = \int d^3v\, m_s \mathbf{v} C(f_s, f_{s'})]$ was calculated using Eq. (28). It was shown that the instability-enhanced friction dominates the stable plasma friction within a short distance (a few Debye lengths) whenever two-stream instabilities are present. In particular, $\mathbf{R}_{\mathrm{IE}}^{1-2}/\mathbf{R}_{\mathrm{LB}}^{1-2} \sim 10^4$ for a wave growth distance of 10 Debye lengths. This distance is much shorter than the presheath length scale. In this application it is also important that momentum be conserved for collisions between individual species. Equation (46) shows that the kinetic theory obeys this property.

The instability-enhanced friction creates a very stiff system where if the relative flow between ion species exceeds a critical value at which the streaming instabilities arise ($\Delta\mathbf{V} = \mathbf{V}_1 - \mathbf{V}_2 \ge \Delta\mathbf{V}_c$), the friction rapidly forces the fluid velocities together and reduces the instability amplitude. Thus, the kinetic theory remains valid as long as the instability amplitude is reduced before reaching nonlinear levels. However, since nonlinear fluctuations would imply a friction over 10^6 times the stable plasma level, the friction has dominated and forced the speeds together well before reaching this amplitude. The implication is that as long as $\Delta V_c < c_{s,1} - c_{s,2}$, instabilities will arise in the presheath and force the relative flow speed of ion species to the critical condition $\Delta V = \Delta V_c$. Otherwise no ion-ion streaming instabilities are expected, and Franklin's solution that each species obtain a speed close to its individual sound speed should hold.

Reference 16 presents a cold ion model for the two-stream instabilities and shows that in this case $\Delta V_c \to 0$. Thus, for cold ion plasmas, Eq. (65) and $\Delta V=0$ predict that the speed of each ion species is the common system sound speed, $V_i = c_s$. This solution is consistent with experimental measurements.[28] It is also noteworthy that two-stream instabilities have been directly measured in the presheath of similar plasmas.[29] Accounting for small finite ion temperatures, such that $\Delta V_c \sim \mathcal{O}(v_{T,i}) \ll c_s$, the condition $\Delta V = \Delta V_c$ and Eq. (65) show that

$$V_1 \simeq c_s + \frac{n_2}{n_e}\frac{c_{s,2}^2}{c_s^2}\Delta V_c \quad \text{and} \quad V_2 \simeq c_s - \frac{n_1}{n_e}\frac{c_{s,1}^2}{c_s^2}\Delta V_c. \qquad (66)$$

Thus, in typical gas discharge plasmas, ions fall into the sheath near the system sound speed c_s rather than their individual sound speeds $c_{s,i}$.

VI. CONCLUSIONS

A kinetic equation that generalizes the Lenard–Balescu equation to describe weakly unstable plasmas has been developed using two independent methods: the test-particle approach in Ref. 1 and the BBGKY hierarchy in Sec. II of this work. The resultant collision operator, Eq. (28), can be written in the Landau form with both drag and diffusion terms. It obeys properties such as conservation laws and the Boltzmann $\mathcal{H}$-theorem. We have also shown that, within the weak-instability approximation $\gamma/\omega_R \ll 1$, instability-enhanced collisions can shorten the time scale on which equilibration of the distribution functions to Maxwellians occurs. The instability-enhanced contribution to the total collision operator was shown to have a diffusive form that fits the framework of conventional quasilinear theory, but for which the continuing source of fluctuations is self-consistently accounted for and is due to discrete particle motion. This led to a determination of the spectral energy density that is absent in conventional quasilinear theory.

This kinetic equation connects the work of Kent and Taylor,[10] which introduced the concept that collective fluctuations arise from discrete particle motion, with previous kinetic and quasilinear theories for scattering in weakly unstable plasmas (such as Rogister and Oberman[7]), which treated the fluctuations as independent of the discrete particle motions. Baldwin and Callen[11] also made this connection for the case of loss cone instabilities in magnetic mirror machines and calculated the resultant instability-enhanced scattering.

Here we have considered a general formulation for unmagnetized plasmas. However, the basic result that the collision frequency due to instability-enhanced interactions scales as the product of $\delta/\ln \Lambda$ and the energy amplification due to fluctuations is common to this work and that of Baldwin and Callen.[11] Here δ is typically a small number $\delta \sim 10^{-2}-10^{-3}$, which depends on the fraction of wave-number space that is unstable. Although the theory is limited by the assumption that the fluctuation amplitude be linear, we have found that instabilities can enhance the collision frequency by a few orders of magnitude before nonlinear amplitudes are reached.

The theory has been successfully applied to two outstanding problems: Langmuir's paradox and determining the Bohm criterion in multiple ion species plasmas. It was shown that Langmuir's paradox can be explained by instability-enhanced collisions that arise due to ion-acoustic instabilities in the presheath. In this application, the convective instabilities propagate out of the plasma before reaching nonlinear levels. Thus, the theory is well suited to describe this problem. The Bohm criterion in low ion temperature, multiple ion species plasmas was shown to be determined by instability-enhanced friction between ion species because ion-ion streaming instabilities are present in the plasma-boundary transition (presheath) region. It was found that the speed of each ion species is close to a common system sound speed rather than the commonly accepted result of individual sound speeds. In this application, the linear kinetic theory is valid because the instability-enhanced collisional friction modifies the plasma dielectric to limit the instability amplitude so that nonlinear fluctuation levels are never reached.

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