KINETIC THEORY OF INSTABILITY-ENHANCED COLLECTIVE INTERACTIONS IN PLASMA

By

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To Colleen

Abstract

A kinetic theory for collective interactions that accounts for electrostatic instabilities in unmagnetized plasmas is developed and applied to two unsolved problems in low-temperature plasma physics: Langmuir's paradox and determining the Bohm criterion for multiple-ion-species plasmas. The basic theory can be considered an extension of the Lenard-Balescu equation to include the effects of wave-particle scattering by instability-amplified fluctuations that originate from discrete particle motion. It can also be considered an extension of quasilinear theory that identifies the origin of fluctuations from discrete particle motion. Emphasis is placed on plasmas with convective instabilities that either propagate out of the domain of interest, or otherwise modify the distribution functions to limit the instability amplitude, before nonlinear amplitudes are reached. Specification of the discrete particle origin of fluctuations allows one to show properties of the resultant collision operator that cannot be shown from conventional quasilinear theory (which does not specify an origin for fluctuations). Two important properties for the applications that we consider are momentum conservation for collisions between individual species and that instabilities drive each species toward Maxwellian distributions.

Langmuir's paradox refers to a measurement showing anomalous electron scattering rapidly establishing a Maxwellian distribution near the boundary of gas-discharge plasmas with low temperature and pressure. We show that this may be explained by instability-enhanced scattering in the plasma-boundary transition region (presheath) where convective ion-acoustic instabilities are excited. These instability-amplified fluctuations exponentially [$\sim \exp(2\gamma t)$] enhance the electron-electron scattering frequency by more than two orders of magnitude, but convect out of the plasma before reaching nonlinear amplitudes. The Bohm criterion for multiple ion species is a single condition that ion flow speeds must obey at the sheath edge; but it is insufficient to determine the flow speed of individual species. We show that an instability-enhanced collisional friction, due to ion-ion streaming instabilities in the presheath, determines this criterion. In this case the strong frictional force modifies the equilibrium, which reduces the instability growth rate and limits the instability amplitude to a low enough level that the basic theory remains valid.

Acknowledgments

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The technical aspects of this work were done in collaboration with Professors Chris Hegna and Jim Callen. Prof. Hegna was my graduate academic advisor. He, more than anyone else, taught me how to approach research in theoretical physics. I have benefited greatly by drawing from his vast stores of plasma theory knowledge, which is impressively broad. More importantly, though, he has taught me how to approach research problems that often seem daunting at the outset by applying a variety of approaches and to *keep calculating* (even if it means bringing a pad of paper to seminars). It turned out that even though my original approaches rarely worked, we could often learn enough from the reasons why they didn't that we could formulate working theories after enough iterations. I'm also thankful to Prof. Hegna for allowing me significant independence in choosing research topics. Although he (and Prof. Callen) often led our research discussions, Prof. Hegna kept my education as his highest priority by requiring that I do "the heavy lifting" on my own. This was essential for my ability to get excited about discovering something and for getting me to the point where I can independently identify and carry out research in theoretical physics. For this, I will be forever grateful.

Like Prof. Hegna, Prof. Callen was careful not to let his superior knowledge of plasma kinetic theory intimidate me during our meetings together (although his 500 page Ph.D. dissertation is intimidating). He acted largely as a second advisor to me and I am immensely grateful to him for spending so much time doing research with me even though he is retired. I am especially thankful for his comprehensive set of lecture notes on plasma kinetic theory and draft textbook copy, which were instrumental in carrying out this work. The kinetic theory review that is presented in Chapter 1 of this dissertation is largely

based on these sources, although other sources are cited because Prof. Callen's have not been published. I will forever remember Prof. Callen's notoriously detailed edits of manuscripts. These usually started with a complimentary message about how the writing looked good, and that his own draft papers have a lot of edits too, followed by a manuscript that is nearly dripping with red ink. However, the edits he provided were always well thought through and their collective effect was always a significantly improved paper. I have learned to be a better writer from Prof. Callen's comments.

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Finally, I'd like to thank the people who funded this research: the American taxpayers. Much is left to be discovered in plasma physics. Although the proponents of our subject have not always been great predictors of the timeline over which we should expect society to benefit from our research, significant advances toward fusion energy reactors continue to be made and industrial processes based on plasmas continue to be discovered that contribute to a broad range of applications including health, communications, lighting and semiconductor manufacturing. Plasma physics truly has the potential to transform the energy and industrial landscape, which is required in order to advance our nation and the world. I hope that it continues to be supported. This work was supported under a National Science Foundation Graduate Research Fellowship and by United States Department of Energy grant numbers DE-FG02-97ER54437 and DE-FG02-86ER53218.

List of Figures

Contents

Chapter 1

Introduction and Background

Plasmas consist of ions and electrons that interact with one another through their self-generated electromagnetic fields and, if present, with externally applied fields. By tracking individual particles, the laws of classical mechanics combined with Maxwell's equations formally provide a complete description of non-relativistic plasma. However, such essentially exact formulations are exceedingly complicated because they require tracking a huge number of particles that all interact with one another simultaneously. For example, the plasma in a fluorescent light bulb contains approximately 10^{10} charged particles (assuming a typical density of 10^{14} m^{-3} and volume of 100 cm^3). Even the fastest supercomputers cannot calculate the evolution of every individual particle in such a complicated system for a meaningful amount of time. Thus, it is necessary to formulate approximate descriptions that describe macroscopic properties of a plasma. A hierarchy of approximations leads to the three leading plasma theories: kinetic, multi-fluid and magnetohydrodynamic (MHD) descriptions. In this work, we will mostly be concerned with kinetic theory, which is the most fundamental of these theories, but we will also discuss multi-fluid equations that can be obtained from the kinetic description.

Rather than describe the position and velocity of every individual particle in a plasma, kinetic theory divides the plasma into different classes of particles (or species) and describes the evolution of the velocity distribution of each species of particles. Species are typically classified by particles with the same charge and mass. Much of modern plasma kinetic theory was introduced by Landau starting in 1936 [1]. Landau first derived his kinetic equation from the small-momentum-transfer-limit of the Boltzmann equation (see section 1.1.3). Landau's equation has proved to be very robust and it is still frequently used today. Only minor modifications have been made to it and these are concerned with more accurately treating physical arguments he made regarding particles interacting in the limits of very short and very long impact parameters (see section 1.1). A theory that self-consistently accounts

for long impact parameters is the Lenard-Balescu equation [2, 3], and theories that properly treat both limits are called convergent kinetic theories. An important physical effect that kinetic theory captures, but which conventional fluid and MHD approximations do not, is Landau damping [4] (although some fluid models such as "gyrofluid" theories account for Landau damping in an approximate manner by including kinetic corrections). Landau damping is a process by which waves can either damp, or grow, in a plasma. Waves can damp or grow by different physical mechanisms in fluid descriptions as well, which are also captured by kinetic theory, but Landau damping is fundamentally a kinetic process.

In stable plasmas all fluctuations damp, often by Landau damping, and scattering is dominated by conventional Coulomb interactions between individual particles. Landau's kinetic equation, as well as the Lenard-Balescu equation, assume that the plasma is stable, but plasmas are not always stable. The presence of a free energy source can cause fluctuations to grow. Growing fluctuations are called instabilities and they are a collective wave motion of the plasma. If an instability amplitude becomes large enough, the scattering of particles can be dominated by the interaction of particles with collective waves, rather than the conventional Coulomb interaction between individual particles.

Theories that describe the scattering of particles from collective wave motion typically assume that the instability amplitude is so large that conventional Coulomb interactions are negligible compared to the wave-particle interactions, while stable plasma theories assume that Coulomb interactions dominate. In this work we consider an intermediate regime: weakly unstable plasmas in which collective fluctuations may be, but are not necessarily, the dominant scattering mechanism and for which the collective fluctuation amplitude is sufficiently weak that nonlinear wave-wave interactions are subdominant. We emphasize convective instabilities that either leave the plasma (or region of interest) or modify the particle distribution functions to limit the fluctuation amplitude before nonlinear amplitudes are reached. We discuss applications for each of these cases and show that collective fluctuations can be the dominant mechanism for scattering particles even when they are in a linear growth regime.

Kinetic equations for weakly unstable plasmas have also been developed by other authors. Frieman and Rutherford [5] used a BBGKY hierarchy approach, but focused on nonlinear aspects such as mode-coupling that enter the kinetic equation at higher order in the hierarchy expansion than we consider in this work. The part of their collision operator that described collisions between particles and collective fluctuations also depended on an initial fluctuation spectrum that must be determined external to the

theory. Rogister and Oberman [6, 7] started from a test particle approach and focused on the linear growth regime, but the fluctuation-induced scattering term in their kinetic equation also depended on specifying an initial fluctuation spectrum external to the theory. Imposing an initial fluctuation spectrum is also a feature of Vlasov theories of fluctuation-induced scattering, such as quasilinear theory [8–10]. These theories can be applied in situations where fluctuations are externally applied to the plasma. In such scenarios, the antenna exciting the waves determines the source fluctuation spectrum. However, fluctuations often originate from within the plasma itself. In this case, the motion of discrete particles creates a source of fluctuations that is not accounted for in these theories. A distinguishing feature of the work presented in this dissertation is that the source fluctuation spectrum, which becomes amplified and leads to instability-enhanced collisions, is self-consistently accounted for by its association with discrete particle motion.

Related work by Kent and Taylor [11] used a WKB method to calculate the amplification of convective fluctuations from discrete particle motion. They focused on describing the fluctuation amplitude, rather than a kinetic equation for particle scattering, and emphasized drift-wave instabilities in magnetized inhomogeneous systems. Baldwin and Callen [12] derived a kinetic equation accounting for the source of fluctuations and their effects on instability-enhanced collisional scattering for the specific case of loss-cone instabilities in magnetic mirror devices.

Our work develops a comprehensive collision operator for unmagnetized plasmas in which electrostatic instabilities that originate from discrete particle motion are accounted for. The resultant collision operator consists of two terms. The first term is the Lenard-Balescu collision operator [2, 3] that describes scattering due to the Coulomb interaction acting between individual particles. The second term is an instability-enhanced collision operator that describes scattering due to collective wave motion. Each term can be written in the Landau form [1], which has both diffusion and drag components in velocity space. The ability to write the collision operator in the Landau form allows proof of physical properties such as the Boltzmann $\mathcal{H}$-theorem and conservation laws for collisions between individual species.

A prominent model used to describe scattering in weakly unstable plasmas is quasilinear theory [8–10]. Quaslinear theory is "collisionless," being based on the Vlasov equation, but has an effective "collision operator" in the form of a diffusion equation that describes wave-particle interactions due to

fluctuations. In the kinetic theory presented here, the instability-enhanced term of the total collision operator for species s, which is a sum of the component collision operators describing collisions of s with each species s', $C(f_s) = \sum_{s'} C(f_s, f_{s'})$, fits into the diffusion equation framework of quasilinear theory. This is because the drag term of the Landau form vanishes in the total collision operator (but not necessarily in the component collision operators). The instability-enhanced contribution to the total collision operator may also be considered an extension of quasilinear theory for the case that instabilities arise within the plasma. Conventional quasilinear theory requires specification of an initial electrostatic fluctuation spectrum by some means external to the theory itself. Our kinetic prescription provides this by self-consistently accounting for the continuing source of fluctuations from discrete particle motion. This determines the spectral energy density of the plasma, which is otherwise an input parameter in conventional quasilinear theory.

We apply the new theory to two unsolved problems in low-temperature plasma physics. The first of these is Langmuir's paradox [13–15], which is a measurement of enhanced electron-electron scattering above the Coulomb level for a stable plasma. We consider the role of instability-enhanced collisions due to ion-acoustic instabilities in the presheath region of Langmuir's discharge and show that they significantly enhance scattering even though the instabilities propagate out of the plasma before reaching nonlinear levels [16]. The second application we consider is determining the Bohm criterion (i.e., the speed at which ions leave a plasma) in plasmas with multiple ion species. In this case, we show that when ion-ion two-stream instabilities arise in the presheath they cause an instability-enhanced collisional friction that is very strong and forces the flow speed of each ion species toward a common speed at the sheath-presheath boundary [17].

The rest of this chapter will proceed in the following manner. A review of previous kinetic equations for stable plasmas is provided in section 1.1. Previous kinetic and Vlasov theories for collisions from wave-particle interactions in unstable plasmas are reviewed in section 1.2, along with a discussion of previous work describing the kinetic (discrete particle) origin of fluctuations. In section 1.3, the utility of the approach taken in this work is discussed. A description of the unsolved problems of Langmuir's paradox and determining the Bohm criterion in multiple-ion-species plasmas are described in sections 1.4 and 1.5, along with a description of how the basic theory developed in this work can be applied to these problems.

The remaining chapters of this dissertation are organized as follows. Our basic kinetic theory for weakly unstable plasmas is derived in chapter 2 using both a dressed test particle approach and the BBGKY hierarchy. In chapter 3, important physical properties of this collision operator are proven and discussed. The connection between it and conventional quasilinear theory is also discussed in this chapter. Chapter 4 presents an application of our basic theory to the Langmuir's paradox problem, where we calculate enhanced electron scattering due to ion-acoustic instabilities in the presheath. In chapter 5 we discuss other kinetic effects in the plasma-boundary transition region, including a kinetic formulation of the Bohm criterion. Finally, in chapter 6, we apply the basic theory to determining the Bohm criterion in plasmas with more than one positive ion species. A brief conclusion follows in chapter 7.

We have also published most of the work presented in this dissertation elsewhere. A derivation of the basic kinetic theory using the dressed test particle method can be found in [18]. The BBGKY hierarchy derivation of this, and its connection to conventional quasilinear theory were presented in [19]. The Langmuir's paradox application and the application of determining the Bohm criterion in multiple-ion-species plasmas were published in [16] and [17].

1.1 Previous Kinetic Theories for Stable Plasmas

In this section, we review the prominent kinetic theories of stable plasmas. These all assume that the dominant scattering mechanism is the Coulomb interaction between individual particles, and ignore scattering from collective fluctuations. They also assume that no equilibrium electric, magnetic, or gravitational fields are applied to the plasma (or, if they are present, that they are weak enough as to not affect the collision operator). General Coulomb scattering theory is reviewed in section 1.1.1, along with a brief derivation of the Boltzmann equation. The Lorentz collision operator is then reviewed in section 1.1.2 by taking the small-momentum-transfer limit of the Boltzmann collision operator. The Lorentz model is the simplest of those presented because it assumes that the plasma consists of only two species, electrons and ions, and it makes two restrictive approximations: that the ions create a stationary background, and that they are infinitely heavy compared to electrons. The Landau collision operator is derived in section 1.1.3, also from the small-momentum-transfer limit of the Boltzmann

equation, but without making the other assumptions of the Lorentz model. In section 1.1.4, it is shown that the Landau collision operator can be written in the same form as the conventional Fokker-Planck equation [20]. This form was first shown in [21], and we refer to it as the Rosenbluth collision operator. The Lenard-Balescu equation is reviewed in section 1.1.5, which accounts for the collective effect of dielectric screening in a plasma that is missed in theories based on the Boltzmann equation. A detailed derivation of the Lenard-Balescu equation is deferred to chapter 2 where it is also generalized to account for unstable plasmas. Finally, in section 1.1.6, convergent collision operators are discussed which account for very small and very large impact parameters. One convergent method is to combine the Boltzmann approach (which captures very small, but not very large, impact parameters) with the Lenard-Balescu equation (which captures very large, but not very small, impact parameters).

1.1.1 The Boltzmann Equation for Coulomb Collisions

The Boltzmann equation achieved great success in the late nineteenth century by accurately describing the kinetics of molecular gases. When the topic of ionized gases was introduced in the early twentieth century, Boltzmann's formalism presented a natural starting point from which to find a plasma kinetic equation. Boltzmann's equation considers only single particle interactions; it assumes that each particle only interacts with one other particle at a time [22]. This assumption is especially good for molecular gases because molecular forces fall off approximately as $\sim r^{-6} - r^{-7}$; hence, the interaction distance is particularly short-range. Charged particles, on the other hand, have Coulomb electric fields that fall off as only r^{-2}; thus one may not expect a single particle interaction approximation to be as good for a plasma. Nevertheless, the Boltzmann's equation has been extended to plasmas and has achieved considerable success in describing basic features of collision processes.

The Boltzmann equation describes the time evolution of the distribution function $f_s(\mathbf{x}, \mathbf{v}, t)$ for a particular species s. The distribution function represents the probable number of particles of species s that will be found at time t in an elemental volume element in the six-dimensional phase-space consisting of physical space d^3x and velocity space d^3v. In a finite time element, dt, the particle coordinates change to

$$\hat{\mathbf{x}} = \mathbf{x} + \mathbf{v}dt \quad \text{and} \quad \hat{\mathbf{v}} = \mathbf{v} + \frac{\mathbf{F}}{m_s}dt \tag{1.1}$$

in which $\mathbf{F}$ is an external force. In the absence of collisions, we would have $f_s(\hat{\mathbf{x}}, \hat{\mathbf{v}}, t + dt) d^3\hat{x}\, d^3\hat{v} = f_s(\mathbf{x}, \mathbf{v}, t) d^3x\, d^3v$. Assuming that $\mathbf{F}$ is an "equilibrium" forcing function, meaning that it is approximately constant on the dt timescale, the Jacobian $\partial(\hat{\mathbf{x}}, \hat{\mathbf{v}})/\partial(\mathbf{x}, \mathbf{v}) = 1$, which implies $d^3\hat{x}\, d^3\hat{v} = d^3x\, d^3v$. With collisions, the distribution function can change over dt, so

$$f_s\left(\mathbf{x} + \mathbf{v}dt, \mathbf{v} + \frac{\mathbf{F}}{m_s}dt, t + dt\right) d^3x\, d^3v - f_s(\mathbf{x}, \mathbf{v}, t) d^3x\, d^3v = \left(\frac{\partial f_s(\mathbf{x}, \mathbf{v}, t)}{\partial t}\right)_{\text{coll}} d^3x\, d^3v\, dt. \tag{1.2}$$

Dividing equation 1.2 by $d^3x\, d^3v\, dt$ and taking the limit $dt \to 0$ yields the Boltzmann equation

$$\boxed{\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} + \frac{\mathbf{F}}{m_s} \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \left(\frac{\partial f_s}{\partial t}\right)_{\text{coll}} \equiv C_{\text{B}}(f_s).} \tag{1.3}$$

Next, we need to determine the collision operator $C_{\text{B}}(f_s)$. The Boltzmann collision operator is based on the assumption that a particle interacts with only one other particle at a time, so the total deflection of a particle can be approximated from a sum of two-body collisions. Working in the center of mass frame, the two-body scattering problem takes the geometry of figure 1.1 where $(\mathbf{v}, \hat{\mathbf{v}})$ are the initial and final velocities of the test-particle of species s and $(\mathbf{v}', \hat{\mathbf{v}}')$ are the initial and final velocities of the background particle of species s'. The s' species can be any species in the plasma, including $s = s'$. The position of the center of mass frame (i.e., scattering center) with respect to laboratory coordinates of each particle $(\mathbf{r}_s, \mathbf{r}_{s'})$ is

$$\mathbf{r}_{\text{cm}} = \frac{m_s \mathbf{r}_s + m_{s'} \mathbf{r}_{s'}}{m_s + m_{s'}}. \tag{1.4}$$

With this, we can write the position of each particles as

$$\mathbf{r}_s = \mathbf{r}_{\text{cm}} - \frac{m_{s'}}{m_s + m_{s'}}\mathbf{r} \quad \text{and} \quad \mathbf{r}_{s'} = \mathbf{r}_{\text{cm}} + \frac{m_s}{m_s + m_{s'}}\mathbf{r}, \tag{1.5}$$

in which $\mathbf{r} \equiv \mathbf{r}_s - \mathbf{r}_{s'}$ is the relative position of the particles. Applying the assumption that the equilibrium forcing function $\mathbf{F}$ is constant on the short collision time scale in equation 1.4 yields $d^2\mathbf{r}_{\text{cm}}/dt^2 = 0$. Newton's equations in the laboratory frame can thus be written $m_s d^2\mathbf{r}_s/dt^2 = -\mathbf{f}$ and $m_{s'} d^2\mathbf{r}_s/dt^2 = \mathbf{f}$, in which $\mathbf{f} = \mathbf{f}_{1,2} = -\mathbf{f}_{2,1}$ is assumed to be a conservative central force. Each of these equations of motion imply

$$m_{ss'} \frac{d^2\mathbf{r}}{dt^2} = \mathbf{f} \tag{1.6}$$

in which

$$\boxed{m_{ss'} = \frac{m_s m_{s'}}{m_s + m_{s'}}} \tag{1.7}$$

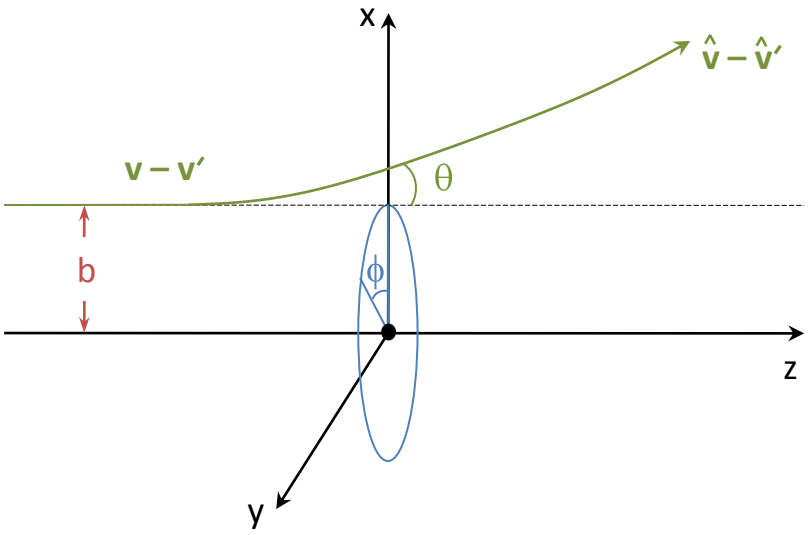

Figure 1.1: Scattering in the center-of-mass reference frame.

is the reduced mass. Equation 1.6 shows that working in the center of mass frame gives the geometry shown in figure 1.1, where the origin is the scattering center (center of mass), θ is the scattering angle and b is the impact parameter, which would be the distance of closest approach if the particles did not interact.

The collision operator represents the change in f_s from particles scattering into the range $(\mathbf{v}, \mathbf{v}+d\mathbf{v})$, balanced by the scattering of particles out of this range,

$$C_{\mathrm{B}}(f_s) = \left(\frac{\partial f_s}{\partial t}\right)_{\mathrm{in}} - \left(\frac{\partial f_s}{\partial t}\right)_{\mathrm{out}} \tag{1.8}$$

The number of particles into an element of area $b\,db\,d\phi$ is $b\,db\,d\phi\,f_s(\mathbf{x},\mathbf{v},t)|\mathbf{v}-\mathbf{v}'|d^3v'$, while the number of background particles in the range $(\mathbf{v}, \mathbf{v}+d\mathbf{v})$ is, by definition, $\sum_{s'} f_{s'}(\mathbf{x},\mathbf{v}',t)d^3v$. Thus, the number of collisions/time within the range $(b, b+db)$ and $(\phi, \phi+d\phi)$ is

$$\sum_{s'} |\mathbf{v}-\mathbf{v}'|\, f_s(\mathbf{x},\mathbf{v},t)\, f_{s'}(\mathbf{x},\mathbf{v}',t)\, b\,db\,d\phi\,d^3v\,d^3v'. \tag{1.9}$$

We will also use the alternate notation

$$b\,db\,d\phi = \frac{d\sigma}{d\Omega}\,d\Omega \tag{1.10}$$

in which $d\sigma/d\Omega$ is the differential scattering cross section, and the solid angle is $d\Omega = \sin\theta\, d\theta\, d\phi$. The rate of change of f_s due to collisions that scatter particles out of the range $(\mathbf{v}, \mathbf{v} + d\mathbf{v})$ is then

$$\left(\frac{\partial f_s}{\partial t}\right)_{\text{out}} = \sum_{s'} \int d^3v' \, |\mathbf{v} - \mathbf{v}'| \int d\Omega \frac{d\sigma}{d\Omega} f_s(\mathbf{x}, \mathbf{v}, t)\, f_{s'}(\mathbf{x}, \mathbf{v}', t). \tag{1.11}$$

Analogously, the change of f_s due to collisions that scatter particles into the range $(\mathbf{v}, \mathbf{v} + d\mathbf{v})$ is

$$\left(\frac{\partial f_s}{\partial t}\right)_{\text{in}} = \sum_{s'} \int d^3\hat{v}' \, |\hat{\mathbf{v}} - \hat{\mathbf{v}}'| \int d\hat{\Omega} \frac{d\hat{\sigma}}{d\hat{\Omega}} f_s(\mathbf{x}, \hat{\mathbf{v}}, t)\, f_{s'}(\mathbf{x}, \hat{\mathbf{v}}', t). \tag{1.12}$$

By symmetry, $d\hat{\sigma} = d\sigma$. From conservation of momentum (or energy) $|\mathbf{v} - \mathbf{v}'| = |\hat{\mathbf{v}} - \hat{\mathbf{v}}'|$, and $d^3v\, d^3v' = d^3\hat{v}\, d^3\hat{v}'$. By putting equations 1.11 and 1.12 into equation 1.8, we arrive at the Boltzmann collision operator [22]

$$\boxed{C_{\text{B}}(f_s) = \sum_{s'} \int d^3v' \int d\Omega \, \frac{d\sigma}{d\Omega} |\mathbf{v} - \mathbf{v}'| \left[f_s(\mathbf{x}, \hat{\mathbf{v}}, t)\, f_{s'}(\mathbf{x}, \hat{\mathbf{v}}', t) - f_s(\mathbf{x}, \mathbf{v}, t)\, f_{s'}(\mathbf{x}, \mathbf{v}', t) \right].} \tag{1.13}$$

The Boltzmann equation 1.13 assumes that the force acting between particles is central and conservative, but nothing more specific. It is thus quite general and can be applied in diverse areas of physics from molecular collisions to the gravitational interaction of stars. Here it will be applied to the electrostatic interaction between charged particles. The forcing function for the electrostatic interaction is

$$\mathbf{f} = q_s q_{s'} \frac{\mathbf{r}}{r^3}, \tag{1.14}$$

where we recall $\mathbf{r} \equiv \mathbf{x} - \mathbf{x}'$. Thus, the equations of motion in the center of mass frame (equation 1.6) can be written

$$m_{ss'} \frac{d\mathbf{u}}{dt} = q_s q_{s'} \frac{\mathbf{r}}{r^3} \tag{1.15}$$

in which

$$\mathbf{u} \equiv \mathbf{v} - \mathbf{v}' \tag{1.16}$$

and we use the notation $r \equiv |\mathbf{r}|$. By definition, the center of mass velocity

$$\mathbf{u}_{\text{cm}} = \frac{m_s \mathbf{v} + m_{s'} \mathbf{v}'}{m_s + m_{s'}} \tag{1.17}$$

is constant in time

$$\frac{d\mathbf{u}_{\text{cm}}}{dt} = \frac{1}{m_s + m_{s'}} \left(m_s \frac{d\mathbf{v}}{dt} + m_{s'} \frac{d\mathbf{v}'}{dt} \right) = \frac{1}{m_s + m_{s'}} \big(\mathbf{f} - \mathbf{f} \big) = 0. \tag{1.18}$$

The velocity vectors of each particle after the collision can be written as

$$\hat{\mathbf{v}} = \mathbf{v} + \Delta\mathbf{v} \quad \text{and} \quad \hat{\mathbf{v}}' = \mathbf{v}' + \Delta\mathbf{v}'. \tag{1.19}$$

Thus, conservation of momentum, $m_s\hat{\mathbf{v}} + m_{s'}\hat{\mathbf{v}}' = m_s\mathbf{v} + m_{s'}\mathbf{v}'$ implies

$$\Delta\mathbf{v}' = -\frac{m_s}{m_{s'}}\Delta\mathbf{v} \quad \text{and} \quad \Delta\mathbf{v} = \frac{m_{ss'}}{m_s}\Delta\mathbf{u}. \tag{1.20}$$

Because of the long interaction distance that results from the $1/r^2$ dependence of the Coulomb interaction, most scattering events produce small-angle collisions. Thus, we expand $f_s(\hat{\mathbf{v}})$ and $f_{s'}(\hat{\mathbf{v}}')$ in equation 1.13 in a Taylor series assuming that $\mathbf{v} \ll \Delta\mathbf{v}$. This yields

$$f_s(\hat{\mathbf{v}}) = f_s(\mathbf{v}) + \Delta\mathbf{v} \cdot \frac{\partial f_s(\mathbf{v})}{\partial \mathbf{v}} + \frac{1}{2}\Delta\mathbf{v}\Delta\mathbf{v} \colon \frac{\partial^2 f_s(\mathbf{v})}{\partial\mathbf{v}\partial\mathbf{v}} + \mathcal{O}\big(\Delta\mathbf{v}\Delta\mathbf{v}\Delta\mathbf{v}\big) \tag{1.21}$$

and

$$f_{s'}(\hat{\mathbf{v}}') = f_{s'}(\mathbf{v}') - \frac{m_s}{m_{s'}}\Delta\mathbf{v} \cdot \frac{\partial f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'} + \frac{1}{2}\frac{m_s^2}{m_{s'}^2}\Delta\mathbf{v}\Delta\mathbf{v} \colon \frac{\partial^2 f_{s'}(\mathbf{v}')}{\partial\mathbf{v}'\partial\mathbf{v}'} + \mathcal{O}\big(\Delta\mathbf{v}\Delta\mathbf{v}\Delta\mathbf{v}\big), \tag{1.22}$$

in which we have written $\Delta\mathbf{v}'$ in terms of $\Delta\mathbf{v}$ using equation 1.20. Putting equations 1.21 and 1.22 into equation 1.13, the small-momentum-transfer limit of the Boltzmann collision operator can be written

$$\begin{aligned} C_{\mathrm{B}}(f_s) \approx \sum_{s'} \int d^3v'\, u \int d\Omega \frac{d\sigma}{d\Omega} \Bigg[& f_{s'}(\mathbf{v}')\Delta\mathbf{v} \cdot \frac{\partial f_s(\mathbf{v})}{\partial \mathbf{v}} - \frac{m_s}{m_{s'}} f_s(\mathbf{v})\Delta\mathbf{v} \cdot \frac{\partial f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'} \\ & - \frac{m_s}{m_{s'}}\Delta\mathbf{v}\Delta\mathbf{v} \colon \frac{\partial f_s(\mathbf{v})}{\partial \mathbf{v}} \frac{\partial f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'} + \frac{1}{2} f_{s'}(\mathbf{v}')\Delta\mathbf{v}\Delta\mathbf{v} \colon \frac{\partial^2 f_s(\mathbf{v})}{\partial\mathbf{v}\partial\mathbf{v}} + \frac{1}{2} f_s(\mathbf{v}) \frac{m_s^2}{m_{s'}^2}\Delta\mathbf{v}\Delta\mathbf{v} \colon \frac{\partial^2 f_{s'}(\mathbf{v}')}{\partial\mathbf{v}'\partial\mathbf{v}'} \Bigg], \end{aligned} \tag{1.23}$$

in which we use the notation $u \equiv |\mathbf{v} - \mathbf{v}'|$.

Equation 1.23 can be simplified by integrating by parts the terms with $\partial/\partial\mathbf{v}'$ derivatives. For example, the first of these terms can be written

$$\int d^3v'\, u \int d\sigma\, \Delta\mathbf{v} \cdot \frac{\partial f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'} = \underbrace{\int d^3v' \frac{\partial}{\partial \mathbf{v}'} \cdot \left[f_{s'}(\mathbf{v}') \int d\sigma\, \Delta\mathbf{v}\, u \right]}_{=0} - \int d^3v'\, f_{s'}(\mathbf{v}') \frac{\partial}{\partial \mathbf{v}'} \cdot \int d\sigma\, \Delta\mathbf{v}\, u. \tag{1.24}$$

Then, using the fact that

$$\frac{\partial}{\partial \mathbf{v}'} \cdot \int d\sigma\, \Delta\mathbf{v}\, u = -\frac{\partial}{\partial \mathbf{v}} \cdot \int d\sigma\, \Delta\mathbf{v}\, u, \tag{1.25}$$

equation 1.24 can be written

$$\int d^3v'\, u \int d\sigma\, \Delta\mathbf{v} \cdot \frac{\partial f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'} = \frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v'\, u\, f_{s'}(\mathbf{v}') \int d\sigma\, \Delta\mathbf{v}. \tag{1.26}$$

Likewise, the third and fifth terms of equation 1.23 can be written

$$\int d^3v'\, u \int d\sigma\, \Delta\mathbf{v}\Delta\mathbf{v} \colon \frac{\partial f_s(\mathbf{v})}{\partial \mathbf{v}} \frac{\partial f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'} = \frac{\partial f_s}{\partial \mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v'\, u\, f_{s'}(\mathbf{v}') \int d\sigma\, \Delta\mathbf{v}\Delta\mathbf{v} \tag{1.27}$$

and

$$\int d^3v'\, u \int d\sigma\, \Delta\mathbf{v}\Delta\mathbf{v} \colon \frac{\partial^2 f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'\mathbf{v}'} = \frac{\partial^2}{\partial \mathbf{v}\partial \mathbf{v}} \colon \int d^3v'\, u\, f_{s'}(\mathbf{v}') \int d\sigma\, \Delta\mathbf{v}\Delta\mathbf{v}. \tag{1.28}$$

Putting equations 1.26, 1.27 and 1.28 into equation 1.23, gives the following expression for the small momentum transfer limit of the Boltzmann collision operator:

$$\boxed{\begin{aligned} C_{\mathrm{B}}(f_s) = \sum_{s'} \Bigg\{ & \left[\frac{\partial f_s}{\partial \mathbf{v}} \cdot - \frac{m_s}{m_{s'}} f_s(\mathbf{v}) \frac{\partial}{\partial \mathbf{v}} \cdot \right] \frac{\langle \Delta\mathbf{v} \rangle^{s/s'}}{\Delta t} \\ & + \left[\frac{1}{2} \frac{\partial^2 f_s(\mathbf{v})}{\partial \mathbf{v}\partial \mathbf{v}} \colon + \frac{1}{2} \frac{m_s^2}{m_{s'}^2} f_s(\mathbf{v}) \frac{\partial^2}{\partial \mathbf{v}\partial \mathbf{v}} \colon - \frac{m_s}{m_{s'}} \frac{\partial f_s(\mathbf{v})}{\partial \mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{v}} \cdot \right] \frac{\langle \Delta\mathbf{v}\Delta\mathbf{v} \rangle^{s/s'}}{\Delta t} \Bigg\} \end{aligned}} \tag{1.29}$$

in which we have defined

$$\frac{\langle \Delta\mathbf{v} \rangle^{s/s'}}{\Delta t} \equiv \int d^3v'\, f_{s'}(\mathbf{v}')\, u \int d\Omega \frac{d\sigma}{d\Omega} \frac{m_{ss'}}{m_s}\, \Delta\mathbf{u} \tag{1.30}$$

and

$$\frac{\langle \Delta\mathbf{v}\Delta\mathbf{v} \rangle^{s/s'}}{\Delta t} \equiv \int d^3v'\, f_{s'}(\mathbf{v}')\, u \int d\Omega \frac{d\sigma}{d\Omega} \frac{m_{ss'}^2}{m_s^2} \Delta\mathbf{u}\Delta\mathbf{u}. \tag{1.31}$$

In equations 1.30 and 1.31, we have written $\Delta\mathbf{v}$ in terms of $\Delta\mathbf{u}$ by applying equation 1.20. If we can find an explicit expression for equations 1.30 and 1.31, equation 1.29 will provide a usable collision operator for a plasma.

One way to approach evaluating equations 1.30 and 1.31 is to find $\Delta\mathbf{u}$ from equation 1.15, which upon integrating, gives

$$m_{ss'} \Delta\mathbf{u} = q_s q_{s'} \int_{-\infty}^{\infty} dt\, \frac{\mathbf{r}}{r^3}, \tag{1.32}$$

where $t = 0$ is set as the time at the distance of closest approach. From the geometry shown in figure 1.2, the position vector is $\mathbf{r} = b(\hat{x}\, \cos\phi + \hat{y}\, \sin\phi) + ut\hat{z}$, thus $r = \sqrt{b^2 + u^2t^2}$. Equation 1.32 thus implies

$$\Delta\mathbf{u}_\perp = \frac{q_s q_{s'}}{m_{ss'}} \int_{-\infty}^{\infty} dt \frac{b\,(\hat{x}\, \cos\phi + \hat{y}\, \sin\phi)}{(b^2 + u^2t^2)^{3/2}} = \frac{2 q_s q_{s'}}{m_{ss'}\, u\, b} \big(\hat{x}\, \cos\phi + \hat{y}\, \sin\phi\big). \tag{1.33}$$

Energy conservation, $m_{ss'}u^2/2 = m_{ss'}|\mathbf{u} + \Delta\mathbf{u}|^2/2 = m_{ss'}(u^2 + 2\mathbf{u}\cdot\Delta\mathbf{u} + \Delta u^2)/2$, implies

$$\mathbf{u} \cdot \Delta\mathbf{u} = -\frac{1}{2} \Delta\mathbf{u} \cdot \Delta\mathbf{u}. \tag{1.34}$$

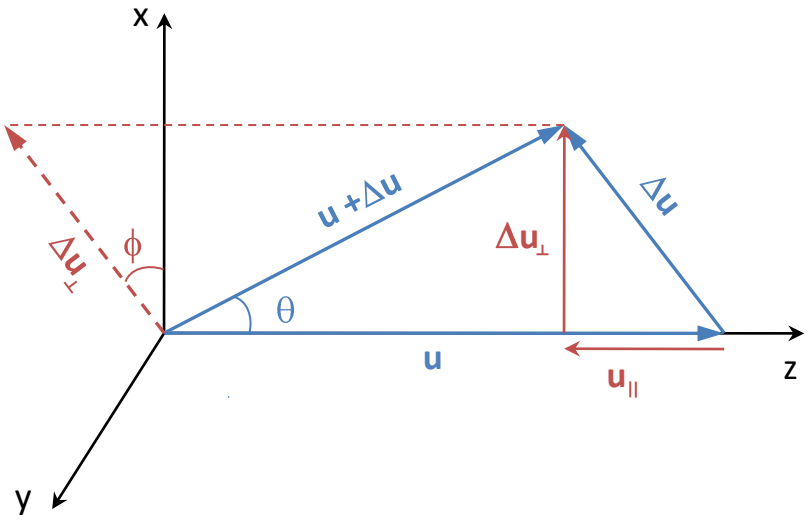

Figure 1.2: Relative velocity vectors after a scattering event in the center of mass frame.

The small angle scattering approximation implies that $\Delta\mathbf{u}_\parallel \ll \Delta\mathbf{u}_\perp$, thus

$$\mathbf{u}\cdot\Delta\mathbf{u} = -\frac{1}{2}\Delta\mathbf{u}\cdot\Delta\mathbf{u} = -\frac{1}{2}\left(\Delta\mathbf{u}_\perp\cdot\Delta\mathbf{u}_\perp + \Delta\mathbf{u}_\parallel\cdot\Delta\mathbf{u}_\parallel\right) \approx -\frac{1}{2}\Delta\mathbf{u}_\perp\cdot\Delta\mathbf{u}_\perp \tag{1.35}$$

and

$$\Delta\mathbf{u}_\parallel = \frac{\mathbf{u}\cdot\Delta\mathbf{u}}{u}\frac{\mathbf{u}}{u} \approx -\frac{1}{2}\frac{\Delta\mathbf{u}_\perp\cdot\Delta\mathbf{u}_\perp}{u}\frac{\mathbf{u}}{u} = -\frac{2q_s^2q_{s'}^2}{m_{ss'}^2\,u^3\,b^2}\frac{\mathbf{u}}{u}. \tag{1.36}$$

With equations 1.33 and 1.36, equations 1.30 and 1.31 can be evaluated. To do so, we go back to the notation $d\Omega\,d\sigma/d\Omega = b\,db\,d\phi$. In the $\langle\Delta\mathbf{v}\rangle/\Delta t$ term, only the $\Delta\mathbf{u}_\parallel$ term contributes and yields

$$\frac{\langle\Delta\mathbf{v}\rangle^{s/s'}}{\Delta t} = -\frac{m_s}{m_{ss'}}\int d^3v'\,f_{s'}(\mathbf{v}')\frac{\mathbf{u}}{u^3}\left(\frac{4\pi q_s^2q_{s'}^2}{m_s^2}\int\frac{db}{b}\right). \tag{1.37}$$

Conversely, only $\Delta\mathbf{u}_\perp$ will contribute to the $\langle\Delta\mathbf{v}\Delta\mathbf{v}\rangle/\Delta t$ term because we enforce the approximaton

$$\Delta\mathbf{u}\,\Delta\mathbf{u} \approx \Delta\mathbf{u}_\perp\Delta\mathbf{u}_\perp = \frac{4q_s^2q_{s'}^2}{m_{ss'}^2\,u^2\,b^2}\begin{pmatrix}\cos^2\phi & \cos\phi\,\sin\phi & 0\\ \cos\phi\,\sin\phi & \sin^2\phi & 0\\ 0 & 0 & 0\end{pmatrix} \tag{1.38}$$

Carrying out the integrals gives

$$\frac{\langle\Delta\mathbf{v}\,\Delta\mathbf{v}\rangle^{s/s'}}{\Delta t} = \int d^3v'\,f_{s'}(\mathbf{v}')\frac{u^2\mathcal{I}-\mathbf{uu}}{u^3}\left(\frac{4\pi q_s^2q_{s'}^2}{m_s^2}\int\frac{db}{b}\right). \tag{1.39}$$

In equations 1.37 and 1.39, we have deliberately not specified the limits of integration in the db integral. Nominally, this integral should range over all values; $b : 0 \to \infty$. However, if these limits of integration are imposed the integral diverges for *both* the large and small b limits: $\ln(\infty/0) \to \infty$. The divergence for large b is a consequence of the fact that the Boltzmann equation assumes that particles only interact as two-body collisions. This results in neglect of collective effects because particles interact with one another according to the $1/r$ potential. However, in a plasma polarization causes dielectric screening of a charged particle, so the potential really scales as $\phi \sim \exp(-r/\lambda_D)/r$. Debye shielding implies that the electrostatic interaction between particles that are more than a Debye length apart is so weak that the particles effectively do not interact. This suggests that the $\int db/b$ integral for large b should be truncated by

$$\boxed{b_{\max} = \lambda_D} \tag{1.40}$$

because of Debye shielding. The Lenard-Balescu formulation for a kinetic equation self-consistently accounts for collective effects, and thus captures Debye shielding. The more rigorous Lenard-Balescu result in theory can give a more complicated equation for $b_{\max}$ than equation 1.40, but for essentially all applications it confirms that equation 1.40 is appropriate. In any case, the overall corrections are only logarithmically dependent on $b_{\max}$ and thus small corrections have a negligible effect on the collision operator.

The lower limit cutoff ($b_{\min}$) should be accounted for by the Boltzmann equation because it can describe large-angle scattering. However, it was not accounted for here because we later assumed small-angle scattering through the approximation in equation 1.35. The $b_{\min}$ can be more rigorously accounted for with the Rutherford scattering formula, which we show next, but a simple physical argument can also provide a good estimate of $b_{\min}$. We expect large angle scattering (near 90°) when the electrostatic potential energy for two particles interacting, $|\mathbf{x}-\mathbf{x}'| = b_{\min}$, is approximately twice the average kinetic energy $m_{ss'}\bar{u^2}/2$, which gives

$$\boxed{b_{\min}^{\mathrm{cl}} = \frac{q_s\, q_{s'}}{m_{ss'}\bar{u^2}}.} \tag{1.41}$$

Alternatively, quantum mechanical effects can induce large-angle scattering when $b_{\min}$ approaches the de Broglie wavelength $\lambda_h/2\pi \approx h/(2\pi m_{ss'}\sqrt{\bar{u^2}})$. Setting $b_{\min} = (\lambda_h/2\pi)/2$ gives a quantum-mechanical

estimate for $b_{\min}$

$$\boxed{b_{\min}^{\rm qm} = h/(4\pi m_{ss'}\sqrt{\bar{u^2}}).} \tag{1.42}$$

In this work we will only be concerned with the classical case of equation 1.41, but in general one should use [22]

$$b_{\min} = \max\{b_{\min}^{\rm cl}, b_{\min}^{\rm qm}\}. \tag{1.43}$$

With the understanding that the limits of integration are physically limited, the b integral in equations 1.37 and 1.39 can be evaluated

$$\int \frac{db}{b} = \int_{b_{\min}}^{b_{\max}} \frac{db}{b} = \ln\left(\frac{b_{\max}}{b_{\min}}\right) = \ln\Lambda_{ss'} \tag{1.44}$$

in which $\Lambda_{ss'} = b_{\max}/b_{\min}$ is the Coulomb logarithm. One final assumption is also typically made, which is to neglect the v' dependence inside the Coulomb logarithm and take for the average energy $\bar{u}$ an average thermal speed $\bar{u}^2 \approx v_{T,ss'}^2 = v_{Ts}^2 + v_{Ts'}^2$, in which $v_{Ts}^2 = 2T_s/m_s$ and T_s is the temperature. This approximation is based upon the fact that the v' variable is integrated in equations 1.30 and 1.31, where the integrand is proportional to $f_s(\mathbf{v}')$, so the characteristic speed of this integral is v_{Ts}. However, it may be possible to find an example where this is not a good approximation (for example if there is a very fast flow). In such a case the Coulomb logarithm may be modified. We will not be concerned with finding such corrections in this work. As a testament to the robustness of this approximation, it should also be noted that significant corrections to the Coulomb logarithm are rarely found in conventional plasmas.

With the identifications above, equation 1.37 is

$$\boxed{\frac{\langle\Delta\mathbf{v}\rangle^{s/s'}}{\Delta t} = -\frac{m_s}{m_{ss'}}\Gamma_{ss'}\int d^3v'\, f_{s'}(\mathbf{v}')\frac{\mathbf{u}}{u^3} = \Gamma_{ss'}\frac{\partial H_{s'}(\mathbf{v})}{\partial\mathbf{v}}} \tag{1.45}$$

and equation 1.39 is

$$\boxed{\frac{\langle\Delta\mathbf{v}\,\Delta\mathbf{v}\rangle^{s/s'}}{\Delta t} = \Gamma_{ss'}\int d^3v'\, f_{s'}(\mathbf{v}')\frac{u^2\mathcal{I} - \mathbf{uu}}{u^3} = \Gamma_{ss'}\frac{\partial^2 G_{s'}(\mathbf{v})}{\partial\mathbf{v}\,\partial\mathbf{v}}.} \tag{1.46}$$

In equations 1.45 and 1.46, we have defined

$$\boxed{\Gamma_{ss'} \equiv \frac{4\pi q_s^2 q_{s'}^2}{m_s^2}\ln\Lambda_{ss'},} \tag{1.47}$$

and the functions $H_{s'}$ and $G_{s'}$ are the Rosenbluth potentials

$$H_{s'}(\mathbf{v}) \equiv \left(1 + \frac{m_s}{m_{s'}}\right) \int d^3v' \frac{f_{s'}(\mathbf{v}')}{|\mathbf{v} - \mathbf{v}'|} \tag{1.48}$$

and

$$G_{s'} \equiv \int d^3v' f_{s'}(\mathbf{v}')|\mathbf{v} - \mathbf{v}'|. \tag{1.49}$$

It will be useful to work with the Rosenbluth potentials throughout this work, and their properties are summarized in appendix A. Putting equations 1.45 and 1.46 into equation 1.29 yields a complete collision operator that describes Coulomb interactions in a stable plasma.

A second way to obtain equations 1.45 and 1.46 that can self-consistently capture $b_{\min}$, because it does not depend on the small angle approximation of equation 1.35, is based on carrying out the integrals of equations 1.30 and 1.31 using the Rutherford scattering formula for the differential cross section in a Coulomb scattering event

$$\frac{d\sigma}{d\Omega} = \frac{q_s^2\, q_{s'}^2}{4m_{ss'}^2\, u^4} \frac{1}{\sin^4\!\big(\theta/2\big)}. \tag{1.50}$$

A derivation of the Rutherford scattering formula, first derived in [23], can be found in essentially any classical mechanics book, see for example [24], so we do not repeat the derivation here.

From the geometry of figure 1.2, $\mathbf{u} + \Delta\mathbf{u} = u[\sin\theta\, \cos\phi\, \hat{x} + \sin\theta\, \sin\phi\, \hat{y} + \cos\theta\, \hat{z}]$. Thus, using the identity $\cos\theta - 1 = -2\sin^2(\theta/2)$, gives

$$\Delta\mathbf{u} = u\big[\sin\theta\, \cos\phi\, \hat{x} + \sin\theta\, \sin\phi\, \hat{y} - 2\sin^2(\theta/2)\, \hat{z}\big]. \tag{1.51}$$

Putting equation 1.51 into equation 1.30 with $d\Omega = \sin\theta\, d\theta d\phi$, yields

$$\frac{\langle \Delta\mathbf{v} \rangle^{s/s'}}{\Delta t} = \frac{q_s^2 q_{s'}^2}{4m_s m_{ss'}} \int d^3v' \frac{f_{s'}(\mathbf{v}')}{u^2} \int d\theta \frac{\sin\theta}{\sin^4(\theta/2)} \int_0^{2\pi} d\phi \big[\sin\theta\, \cos\phi\, \hat{x} + \sin\theta\, \sin\phi\, \hat{y} - 2\sin^2(\theta/2)\, \hat{z}\big] \tag{1.52}$$

$$= -\frac{\pi q_s^2 q_{s'}^2}{m_s m_{ss'}} \int d^3v' \frac{f_{s'}(\mathbf{v}')}{u^2} \int d\theta \frac{\sin\theta}{\sin^2(\theta/2)} \hat{z}.$$

Likewise, putting equation 1.51 into equation 1.31 and also evaluating the $d\phi$ integral, yields

$$\frac{\langle \Delta\mathbf{v}\Delta\mathbf{v} \rangle^{s/s'}}{\Delta t} = \frac{\pi q_s^2 q_{s'}^2}{4m_s^2} \int d^3v' \frac{f_{s'}(\mathbf{v}')}{u} \int d\theta \frac{\sin\theta}{\sin^4(\theta/2)} \begin{pmatrix} \sin^2\theta & 0 & 0 \\ 0 & \sin^2\theta & 0 \\ 0 & 0 & 8\sin^4(\theta/2) \end{pmatrix}. \tag{1.53}$$

Again, we have deliberately not specified the limits of integration for the θ integral because we expect it do diverge for long range (large b, small θ) collisions. Thus, we use for the lower limit of integration $\theta_{\rm min}$. Using the Rutherford equation 1.50 accounts for the large angle scattering; thus we can still take $\theta_{\rm max} = \pi$ (which is equivalent to $b_{\rm min} = 0$). In equation 1.52, we require the integral

$$\int_{\theta_{\rm min}}^{\pi} d\theta \frac{\sin\theta}{\sin^2(\theta/2)} = -4\,\ln\left[\sin\left(\frac{\theta_{\rm min}}{2}\right)\right]. \tag{1.54}$$

Similarly, the $\hat{x}\hat{x}$ and $\hat{y}\hat{y}$ components of equation 1.53 require the integral

$$\int_{\theta_{\rm min}}^{\pi} d\theta \frac{\sin^3\theta}{\sin^4(\theta/2)} = -8\,\cos^2\left(\frac{\theta_{\rm min}}{2}\right) - 16\,\ln\left[\sin\left(\frac{\theta_{\rm min}}{2}\right)\right] \approx -16\,\ln\left[\sin\left(\frac{\theta_{\rm min}}{2}\right)\right] \tag{1.55}$$

in which we have anticipated that $\left|\ln[\sin(\theta_{\rm min}/2)]\right| \gg 1$ in the last step.

Next, we determine $\theta_{\rm min}$. From above, we argued that $b_{\rm max} \approx \lambda_D$, due to Debye screening. So, we relate $\theta_{\rm min}$ to $b_{\rm max}$ by putting the Rutherford formula into the geometric relation $b\,db\,d\phi = (d\sigma/d\Omega)\,d\Omega$, and integrating

$$\int_0^{b_{\rm max}} db\,b = \frac{q_s^2 q_{s'}}{4m_{ss'}^2\,u^4}\int_{\theta_{\rm min}}^{\pi} d\theta\,\frac{\sin\theta}{\sin^4\!\big(\theta/2\big)} = \frac{q_s^2 q_{s'}^2}{4m_{ss'}^2 u^4}\left[\frac{2}{\sin^2\!\big(\theta_{\rm min}/2\big)} - 2\right]. \tag{1.56}$$

Rearranging this result gives

$$\sin^2\left(\frac{\theta_{\rm min}}{2}\right) = \frac{1}{1 + (m_{ss'}^2 u^4 b_{\rm max}^2)/(q_s^2 q_{s'}^2)} \approx \frac{q_s^2 q_{s'}^2}{m_{ss'}^2\,u^4\,b_{\rm max}^2}. \tag{1.57}$$

Thus, putting in $b_{\rm max} = \lambda_D$, we find

$$\sin\left(\frac{\theta_{\rm min}}{2}\right) \approx \frac{q_s q_{s'}}{m_{ss'}\,u^2\lambda_D} = \frac{b_{\rm min}^{cl}}{b_{\rm max}} = \frac{1}{\Lambda_{ss'}}. \tag{1.58}$$

So, $\ln[\sin(\theta_{\rm min}/2)] = -\ln\Lambda_{ss'}$.

Finally, putting equation 1.58 into equation 1.54 and the result into equations 1.52 and 1.53 gives

$$\frac{\langle\Delta\mathbf{v}\rangle^{s/s'}}{\Delta t} = -\frac{m_s}{m_{ss'}}\Gamma_{ss'}\int d^3v'\,f_{s'}(\mathbf{v}')\,\frac{\mathbf{u}}{u^3}. \tag{1.59}$$

and

$$\frac{\langle\Delta\mathbf{v}\Delta\mathbf{v}\rangle^{s/s'}}{\Delta t} = \Gamma_{ss'}\int d^3v'\frac{f_{s'}(\mathbf{v}')}{u}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\ln\Lambda_{ss'} \end{pmatrix}. \tag{1.60}$$

Assuming $\ln\Lambda_{ss'} \gg 1$, the $1/\ln\Lambda$ term in the $\hat{z}\hat{z}$ position can be neglected and equation 1.60 is approximately

$$\frac{\langle\Delta\mathbf{v}\Delta\mathbf{v}\rangle^{s/s'}}{\Delta t} = \Gamma_{ss'}\int d^3v' f_{s'}(\mathbf{v}')\frac{u^2\mathcal{I}-\mathbf{uu}}{u^3}. \tag{1.61}$$

Equations 1.59 and 1.61 are identical to equations 1.45 and 1.46 that where obtained using the physical arguments for $b_{\min}$. Using the Rutherford scattering formula has provided a firm foundation for the previous heuristic physical argument for $b_{\min}$. Of course, the determination of $\Lambda_{ss'}$ still required external physical arguments to determine $b_{\max}$. However, this limit too can be firmly established using the Lenard-Balescu equation, which is discussed in section 1.1.5. It is also noteworthy that equation 1.60 shows that corrections to the small-angle scattering approximation come about as $\mathcal{O}(1/\ln\Lambda)$. Thus, $\ln\Lambda \gg 1$ is required for the small angle scattering approximation to be valid.

1.1.2 The Lorentz Collision Operator

The Lorentz collision model is a simple starting point that illustrates the basic effects of momentum loss and velocity-space diffusion in plasmas. It assumes that the plasma consists of a single species of positively charged ions and a single species of negatively charged electrons such that the ions are infinitely heavy and stationary. The Lorentz collision operator then seeks to determine how the electron distribution function evolves due to collisions with the stationary, infinitely heavy, background ion population. The Lorentz approximations can thus be summarized as

$$m_s = m_e, \quad m_{s'} = m_i \to \infty \quad \text{and} \quad f_{s'}(\mathbf{v}) = n_i\delta(\mathbf{v}). \tag{1.62}$$

With these assumptions, equation 1.29 becomes

$$C_{\mathcal{L}}(f_e) = \frac{\partial f_e(\mathbf{v})}{\partial\mathbf{v}}\cdot\frac{\langle\Delta\mathbf{v}\rangle^{e/i}}{\Delta t} + \frac{1}{2}\frac{\partial^2 f_e(\mathbf{v})}{\partial\mathbf{v}\partial\mathbf{v}}:\frac{\langle\Delta\mathbf{v}\Delta\mathbf{v}\rangle^{e/i}}{\Delta t}. \tag{1.63}$$

Expanding the second term gives

$$\frac{1}{2}\frac{\partial^2 f_e}{\partial\mathbf{v}\partial\mathbf{v}}:\frac{\langle\Delta\mathbf{v}\Delta\mathbf{v}\rangle^{e/i}}{\Delta t} = \frac{1}{2}\frac{\partial}{\partial\mathbf{v}}\cdot\left[\frac{\langle\Delta\mathbf{v}\Delta\mathbf{v}\rangle^{e/i}}{\Delta t}\cdot\frac{\partial f_e}{\partial\mathbf{v}}\right] - \frac{1}{2}\frac{\partial f_e}{\partial\mathbf{v}}\cdot\left[\frac{\partial}{\partial\mathbf{v}}\cdot\frac{\langle\Delta\mathbf{v}\Delta\mathbf{v}\rangle^{e/i}}{\Delta t}\right]. \tag{1.64}$$

Recalling from equation A.4 that

$$\frac{\partial}{\partial\mathbf{v}}\cdot\frac{\langle\Delta\mathbf{v}\Delta\mathbf{v}\rangle^{s/s'}}{\Delta t} = 2\frac{m_{ss'}}{m_s}\frac{\langle\Delta\mathbf{v}\rangle^{s/s'}}{\Delta t} \tag{1.65}$$

and that $m_{ss'}/m_s = 1$ in the Lorentz model, then putting equations 1.65 and 1.64 into 1.63, gives

$$C_{\mathcal{L}}(f_e) = \frac{\Gamma_{ei} n_i}{2} \frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{v^2 \mathcal{I} - \mathbf{vv}}{v^3} \cdot \frac{\partial f_e}{\partial \mathbf{v}} \right). \tag{1.66}$$

Noting that $\mathbf{v} \cdot (v^2 \mathcal{I} - \mathbf{vv}) = 0$, the Lorentz collision operator can be written

$$\boxed{C_{\mathcal{L}}(f_e) = \frac{\nu_o}{2} \frac{\partial}{\partial \mathbf{v}} \cdot \left[\left(v^2 \mathcal{I} - \mathbf{vv} \right) \cdot \frac{\partial f_e}{\partial \mathbf{v}} \right]} \tag{1.67}$$

in which

$$\boxed{\nu_o(v) \equiv \frac{4\pi Z_i^2 e^4 n_i}{m_e^2 \, v^3} \ln \Lambda_{ei}} \tag{1.68}$$

is a reference collision frequency. The Lorentz equation can also be written in the convenient notation $C_{\mathcal{L}}(f_e) = \nu_o \mathcal{L}\{f_e(\mathbf{v})\}$ in which

$$\mathcal{L} \equiv \frac{1}{2} \frac{\partial}{\partial \mathbf{v}} \cdot \left(v^2 \mathcal{I} - \mathbf{vv} \right) \cdot \frac{\partial}{\partial \mathbf{v}} = \frac{1}{2} \left(\mathbf{v} \times \frac{\partial}{\partial \mathbf{v}} \right) \cdot \left(\mathbf{v} \times \frac{\partial}{\partial \mathbf{v}} \right). \tag{1.69}$$

1.1.3 The Landau Collision Operator

Landau was the first to apply the small scattering angle approximation to the Boltzmann collision operator (equation 1.13) in order to apply it in a plasma physics context. However, rather than writing the result in the form of equation 1.29, Landau wrote his collision operator in the form of the velocity-space divergence of a collisional current

$$C(f_s) = -\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{J}_v. \tag{1.70}$$

In this section, we show how equation 1.29 can be transformed into this form. In Landau's original work [1], he also introduced the physical arguments made in section 1.1.1 regarding truncation of the logarithmically diverging b integral. His arguments gave equations 1.41 and 1.40 for $b_{\min}$ and $b_{\max}$, which led to determining the Coulomb logarithm.

We seek to write equation 1.29 in the form of a total divergence of the form suggested by equation 1.70. To do so, we will consider each of the five terms of equation 1.29 individually. Using the property

$$\frac{\partial}{\partial \mathbf{v}} \cdot \frac{\langle \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} = -4\pi \, \Gamma_{ss'} \frac{m_s}{m_{ss'}} f_{s'}(\mathbf{v}) \tag{1.71}$$

from equation A.3, and differentiating, the first term can be written as

$$\frac{\partial f_s(\mathbf{v})}{\partial \mathbf{v}} \cdot \frac{\langle \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} = \frac{\partial}{\partial \mathbf{v}} \cdot \left(f_s(\mathbf{v}) \frac{\langle \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} \right) + 4\pi \, \Gamma_{ss'} \frac{m_s}{m_{ss'}} f_s(\mathbf{v}) \, f_{s'}(\mathbf{v}) \tag{1.72}$$

and the second term as

$$\frac{m_s}{m_{s'}} f_s(\mathbf{v}) \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\langle \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} = -\frac{m_s^2}{m_{s'} m_{ss'}} 4\pi \Gamma_{ss'} f_s(\mathbf{v}) f_{s'}(\mathbf{v}). \tag{1.73}$$

Using the property

$$\frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}} : \frac{\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} = -8\pi \Gamma_{ss'} f_{s'}(\mathbf{v}) \tag{1.74}$$

from equation A.5, and differentiating, the third term becomes

$$\frac{1}{2} \frac{\partial^2 f_s}{\partial \mathbf{v} \partial \mathbf{v}} : \frac{\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} = \frac{1}{2} \frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{\partial f_s}{\partial \mathbf{v}} \cdot \frac{\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} \right] - \frac{1}{2} \frac{\partial}{\partial \mathbf{v}} \cdot \left[f_s \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} \right] - 4\pi \Gamma_{ss'} f_s(\mathbf{v}) f_{s'}(\mathbf{v}), \tag{1.75}$$

the fourth term becomes

$$\frac{1}{2} \frac{m_s^2}{m_{s'}^2} f_s(\mathbf{v}) \frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}} : \frac{\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} = -4\pi \frac{m_s^2}{m_{s'}^2} \Gamma_{ss'} f_s(\mathbf{v}) f_{s'}(\mathbf{v}), \tag{1.76}$$

and the fifth, and final, term becomes

$$-\frac{m_s}{m_{s'}} \frac{\partial f_s}{\partial \mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} = -\frac{m_s}{m_{s'}} \frac{\partial}{\partial \mathbf{v}} \cdot \left(f_s(\mathbf{v}) \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} \right) - 8\pi \frac{m_s}{m_{s'}} \Gamma_{ss'} f_s(\mathbf{v}) f_{s'}(\mathbf{v}). \tag{1.77}$$

Plugging equations 1.72 – 1.77 into 1.29 yields

$$C(f_s) = \frac{\partial}{\partial \mathbf{v}} \cdot \sum_{s'} \left[f_s(\mathbf{v}) \frac{\langle \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} + \frac{1}{2} \frac{\partial f_s}{\partial \mathbf{v}} \cdot \frac{\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} - \frac{1}{2} \left(1 + 2 \frac{m_s}{m_{s'}} \right) f_s(\mathbf{v}) \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} \right]. \tag{1.78}$$

Using equation 1.65 to write $\langle \Delta \mathbf{v} \rangle / \Delta t$ in terms of $\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle / \Delta t$, the first and third terms can be combined to give

$$C(f_s) = -\frac{\partial}{\partial \mathbf{v}} \cdot \sum_{s'} \left[\frac{1}{m_{s'}} f_s(\mathbf{v}) \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} - \frac{1}{m_s} \frac{\partial f_s}{\partial \mathbf{v}} \cdot \frac{\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} \right]. \tag{1.79}$$

Inserting equation 1.46 for $\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle / \Delta t$, and using

$$\begin{aligned} \frac{\partial}{\partial \mathbf{v}} \cdot \int d^3 v' f_{s'}(\mathbf{v}') \frac{u^2 \mathcal{I} - \mathbf{u}\mathbf{u}}{u^3} &= \int d^3 v' f_{s'}(\mathbf{v}') \frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{u^2 \mathcal{I} - \mathbf{u}\mathbf{u}}{u^3} \right) \\ &= -\int d^3 v' f_{s'}(\mathbf{v}') \frac{\partial}{\partial \mathbf{v}'} \cdot \left(\frac{u^2 \mathcal{I} - \mathbf{u}\mathbf{u}}{u^3} \right) = \int d^3 v' \frac{\partial f_{s'}}{\partial \mathbf{v}'} \cdot \frac{u^2 \mathcal{I} - \mathbf{u}\mathbf{u}}{u^3} \end{aligned} \tag{1.80}$$

in the first term, yields the Landau form of the collision operator

$$\boxed{C_{\rm L}(f_s) = -\frac{\partial}{\partial \mathbf{v}} \cdot \sum_{s'} \int d^3v' \mathcal{Q}_L \cdot \left(\frac{1}{m_{s'}} \frac{\partial}{\partial \mathbf{v}'} - \frac{1}{m_s} \frac{\partial}{\partial \mathbf{v}} \right) f_s(\mathbf{v}) f_{s'}(\mathbf{v}'),} \tag{1.81}$$

in which

$$\boxed{\mathcal{Q}_{\rm L} \equiv \frac{2\pi q_s^2 q_{s'}^2}{m_s} \ln \Lambda_{ss'} \left(\frac{u^2 \mathcal{I} - \mathbf{uu}}{u^3} \right)} \tag{1.82}$$

is the Landau collisional kernel.

The *total* Landau collision operator, $C_{\rm L}(f_s)$, consists of a sum of component collision operators, $C_{\rm L}(f_s, f_{s'})$, that can each be written as a velocity-divergence of a collisional current

$$\frac{df_s(\mathbf{v})}{dt} = C_{\rm L}(f_s) = \sum_{s'} C_{\rm L}(f_s, f_{s'}) = -\frac{\partial}{\partial \mathbf{v}} \cdot \sum_{s'} \mathbf{J}_{\rm L}^{s/s'} \tag{1.83}$$

in which

$$\mathbf{J}_{\rm L}^{s/s'} = \int d^3v' \mathcal{Q}_{\rm L} \cdot \left(\frac{1}{m_{s'}} \frac{\partial}{\partial \mathbf{v}'} - \frac{1}{m_s} \frac{\partial}{\partial \mathbf{v}} \right) f_s(\mathbf{v}) f_{s'}(\mathbf{v}'). \tag{1.84}$$

1.1.4 The Rosenbluth (Fokker-Planck-like) Collision Operator

In the late 1950's, Rosenbluth, MacDonald and Judd used the Fokker-Planck formalism to derive a kinetic equation for stable plasmas [21]. Their result is commonly called the Fokker-Planck equation for plasmas. The original Fokker-Planck treatment was for molecular gases [20]. The plasma result, which we refer to as the Rosenbluth equation, is equivalent to the Landau collision operator 1.81. In this section we will derive the Rosenbluth form from equation 1.81 and show how it can be written in a form that looks like the classical Fokker-Planck equation.

To show this, we first add and subtract $f_s(\mathbf{v}) \partial f_{s'}(\mathbf{v}')/\partial \mathbf{v}'$ inside the parentheses of the Landau collisional current of equation 1.84. Then it can be written

$$\begin{aligned} \mathbf{J}^{s/s'} = \Gamma_{s,s'} \Bigg\{ & \frac{1}{2} \frac{m_s}{m_{ss'}} \int d^3v' \frac{u^2 \mathcal{I} - \mathbf{uu}}{u^3} \cdot f_s(\mathbf{v}) \frac{\partial f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'} \\ & - \frac{1}{2} \int d^3v' \frac{u^2 \mathcal{I} - \mathbf{uu}}{u^3} \cdot \left[f_{s'}(\mathbf{v}') \frac{\partial f_s(\mathbf{v})}{\partial \mathbf{v}} - f_s(\mathbf{v}) \frac{\partial f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'} \right] \Bigg\}. \end{aligned} \tag{1.85}$$

Considering the first integral, integrating by parts gives

$$\int d^3v' \frac{u^2 \mathcal{I} - \mathbf{uu}}{u^3} \cdot \frac{\partial f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'} = \underbrace{\int d^3v' \frac{\partial}{\partial \mathbf{v}'} \cdot \left[\frac{u^2 \mathcal{I} - \mathbf{uu}}{u^3} f_{s'}(\mathbf{v}') \right]}_{=0} - \int d^3v' f_{s'}(\mathbf{v}') \frac{\partial}{\partial \mathbf{v}'} \cdot \frac{u^2 \mathcal{I} - \mathbf{uu}}{u^3}. \tag{1.86}$$

Using the relations

$$\frac{\partial}{\partial \mathbf{v}'} \cdot \frac{u^2\mathcal{I} - \mathbf{uu}}{u^3} = -\frac{\partial}{\partial \mathbf{v}} \cdot \frac{u^2\mathcal{I} - \mathbf{uu}}{u^3} \tag{1.87}$$

and

$$\frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v' \frac{u^2\mathcal{I} - \mathbf{uu}}{u^3} f_{s'}(\mathbf{v}') = \frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{\partial^2 G_{s'}(\mathbf{v})}{\partial \mathbf{v} \partial \mathbf{v}} \right) = 2\frac{m_{ss'}}{m_s} \frac{\partial H_{s'}(\mathbf{v})}{\partial \mathbf{v}}, \tag{1.88}$$

the first integral can be written in terms of H. Similarly, for the second term we utilize

$$\frac{\partial}{\partial \mathbf{v}} \cdot \left[f_s(\mathbf{v}) \int d^3v' f_{s'}(\mathbf{v}') \frac{u^2\mathcal{I} - \mathbf{uu}}{u^3} \right] = \int d^3v' \frac{u^2\mathcal{I} - \mathbf{uu}}{u^3} \cdot \left[f_{s'}(\mathbf{v}') \frac{\partial f_s(\mathbf{v})}{\partial \mathbf{v}} + f_s(\mathbf{v}) \frac{\partial f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'} \right]. \tag{1.89}$$

Putting these into equation 1.85 leads to the Rosenbluth collision operator

$$\boxed{C_{\mathrm{R}}(f_s) = -\frac{\partial}{\partial \mathbf{v}} \cdot \sum_{s'} \Gamma_{s,s'} \left\{ f_s(\mathbf{v}) \frac{\partial H_{s'}(\mathbf{v})}{\partial \mathbf{v}} - \frac{1}{2} \frac{\partial}{\partial \mathbf{v}} \cdot \left[f_s(\mathbf{v}) \frac{\partial^2 G_{s'}(\mathbf{v})}{\partial \mathbf{v} \partial \mathbf{v}} \right] \right\}.} \tag{1.90}$$

Identifying equations 1.45 and 1.46, this can also be written in a Fokker-Planck form

$$\boxed{C_{\mathrm{FP}}(f_s) = -\frac{\partial}{\partial \mathbf{v}} \cdot \sum_{s'} \left\{ f_s(\mathbf{v}) \frac{\langle \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} - \frac{1}{2} \frac{\partial}{\partial \mathbf{v}} \cdot \left[f_s(\mathbf{v}) \frac{\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} \right] \right\},} \tag{1.91}$$

in which the right side is the sum of a dynamical friction and dispersion.

1.1.5 The Lenard-Balescu Collision Operator

An improvement over the Landau and Fokker-Planck equations has been provided by Lenard and Balescu [2, 3]. The Lenard-Balescu equation accounts for physics of the collective nature of a plasma; thus it accurately accounts for Debye shielding and resolves the $b_{\max}$ (or $k_{\min}$ in Fourier-space) integral self-consistently. It still suffers the logarithmic divergence for hard collisions though, since it makes a small angle collision approximation. A general plasma dielectric function is allowed in the theory, but an adiabatic approximation $\hat{\varepsilon} = 1 + 1/k^2\lambda_D^2$ is often used in practice; in this case the Lenard-Balescu equation reduces to Landau's equation. This is shown below. Chapter 2 extends Lenard-Balescu theory to also include the collective effects of unstable plasmas. Since the Lenard-Balescu equation is easily identified from the more general result that also includes wave-particle interactions, we defer a rigorous derivation of the Lenard-Balescu equation to chapter 2.

The Lenard-Balescu collision operator also has the Landau form

$$C_{\mathrm{LB}}(f_s) = -\frac{\partial}{\partial \mathbf{v}} \cdot \sum_{s'} \int d^3v' \mathcal{Q}_{\mathrm{LB}} \cdot \left(\frac{1}{m_{s'}} \frac{\partial}{\partial \mathbf{v}'} - \frac{1}{m_s} \frac{\partial}{\partial \mathbf{v}} \right) f_s(\mathbf{v}) f_{s'}(\mathbf{v}'), \tag{1.92}$$

but now the collisional kernel is given by

$$\mathcal{Q}_{\rm LB} \equiv \frac{2q_s^2 q_{s'}^2}{m_s} \int d^3k \frac{\mathbf{kk}}{k^4} \frac{\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]}{|\hat{\varepsilon}(\mathbf{k},\mathbf{k}\cdot\mathbf{v})|^2}. \tag{1.93}$$

The Lenard-Balescu equation reduces to the Landau (or, equivalently the Rosenbluth) equation, with the $b_{\rm max} = \lambda_D$ argument applied, if one assumes an adiabatic dielectric response

$$\hat{\varepsilon}(\mathbf{k},\omega) = 1 + \frac{1}{k^2\lambda_{De}^2}. \tag{1.94}$$

Using cylindrical coordinates

$$\begin{aligned} k_x = k_\perp \cos\varphi \ , \quad k_y = k_\perp \sin\varphi \ , \quad k_z = k_\parallel \\ u_x = u_\perp \cos\psi \ , \quad u_y = u_\perp \sin\psi \ , \quad u_z = u_\parallel \end{aligned} \tag{1.95}$$

in which φ is the angle between $k_\perp$ and $\hat{x}$ and ψ is the angle between $u_\perp$ and $\hat{x}$. Then the delta function part can be written

$$\delta(\mathbf{k}\cdot\mathbf{u}) = \delta[k_\perp u_\perp(\cos\varphi\cos\psi + \sin\varphi\sin\psi) + k_\parallel u_\parallel] = \delta[k_\perp u_\perp \cos(\varphi-\psi) + k_\parallel u_\parallel]. \tag{1.96}$$

Equation 1.93 can then be written

$$\begin{aligned} \mathcal{Q} = \frac{2q_s^2 q_{s'}^2}{m_s} \int_0^{2\pi} d\varphi \int_0^\infty dk_\perp \, k_\perp \int_{-\infty}^\infty dk_\parallel \frac{\delta[k_\parallel u_\parallel + k_\perp u_\perp \cos(\varphi-\psi)]}{(k_\parallel^2 + k_\perp^2 + \lambda_{De}^{-2})^2} \\ \times \begin{pmatrix} k_\perp^2\cos^2\varphi & k_\perp^2 \sin\varphi\cos\varphi & k_\perp k_\parallel \cos\varphi \\ k_\perp^2 \sin\varphi\cos\varphi & k_\perp^2\sin^2\varphi & k_\perp k_\parallel \sin\varphi \\ k_\perp k_\parallel \cos\varphi & k_\perp k_\parallel \sin\varphi & k_\parallel^2 \end{pmatrix}. \end{aligned} \tag{1.97}$$

After the $k_\parallel$ integral this is

$$\mathcal{Q} = \frac{2q_s^2 q_{s'}^2}{m_s} \int_0^{2\pi} d\varphi \int_0^\infty dk_\perp \frac{k_\perp^3/u_\parallel}{\left[k_\perp^2\left(1+\frac{u_\perp^2}{u_\parallel^2}\cos^2(\varphi-\psi)\right) + \lambda_{De}^{-2}\right]^2} \mathcal{T} \tag{1.98}$$

in which $\mathcal{T}$ is the tensor

$$\mathcal{T} \equiv \begin{pmatrix} \cos^2\varphi & \sin\varphi\cos\varphi & -\frac{u_\perp}{u_\parallel}\cos(\varphi-\psi)\cos\varphi \\ \sin\varphi\cos\varphi & \sin^2\varphi & -\frac{u_\perp}{u_\parallel}\cos(\varphi-\psi)\sin\varphi \\ -\frac{u_\perp}{u_\parallel}\cos(\varphi-\psi)\cos\varphi & -\frac{u_\perp}{u_\parallel}\cos(\varphi-\psi)\sin\varphi & \frac{u_\perp^2}{u_\parallel^2}\cos^2(\varphi-\psi) \end{pmatrix}. \tag{1.99}$$

Next, consider the the $k_\perp$ integral, which must be truncated for large k at $1/b_{\min}$, and note that

$$\int_0^{1/b_{\min}} dk_\perp \frac{k_\perp^3}{(k_\perp^2 a + \lambda_{De}^{-2})^2} = \frac{1}{2a^2}\big(2\ln\Lambda + \ln a - 1\big) \approx \frac{\ln\Lambda}{a^2} \tag{1.100}$$

in which $a \equiv 1 + u_\perp^2/u_\parallel^2 \cos^2(\varphi - \psi)$ is a number close to unity and $a \ll \Lambda$ where $\Lambda \equiv \lambda_{De}/b_{min}$. The collisional kernel is then

$$\mathcal{Q} = \frac{2q_s^2 q_{s'}^2}{m_s} u_\parallel^3 \int_0^{2\pi} d\varphi \frac{\ln\Lambda}{\big[u_\parallel^2 + u_\perp^2 \cos^2(\varphi - \psi)\big]^2}\,\mathcal{T}. \tag{1.101}$$

Completing the φ integrals gives

$$\mathcal{Q} = \frac{2\pi q_s^2 q_{s'}^2}{m_s} \ln\Lambda \frac{1}{u^3} \begin{pmatrix} u_\parallel^2 + u_\perp^2 \sin^2\psi & -u_\perp^2 \cos\psi \sin\psi & -u_\perp u_\parallel \cos\psi \\ -u_\perp^2 \cos\psi \sin\psi & u_\parallel^2 + u_\perp^2 \cos^2\psi & -u_\perp u_\parallel \sin\psi \\ -u_\perp u_\parallel \cos\psi & -u_\perp u_\parallel \sin\psi & u_\perp^2 \end{pmatrix}, \tag{1.102}$$

which is simply the Landau collisional kernel

$$\mathcal{Q}_{\mathrm{L}} = \frac{2\pi q_s^2 q_{s'}^2}{m_s} \ln\Lambda_{ss'} \frac{u^2\mathcal{I} - \mathbf{uu}}{u^3}. \tag{1.103}$$

1.1.6 Convergent Collision Operators

In the preceding sections, we have seen two fundamentally different approaches to developing a kinetic theory of plasma: the small momentum transfer limit of the Boltzmann equation ($\Delta\mathbf{v} \ll \mathbf{v}$) and the Lenard-Balescu equation based on a perturbation of the distribution itself $\delta f_s \ll f_s$. Both approaches resulted in a kinetic equation that requires truncation of an otherwise divergent integral. This truncation is provided by physical arguments external to the theories themselves. The necessity to do this demonstrates limitations of each theoretical approach. The limitation of the small momentum transfer limit of the Boltzmann equation is that it neglects Debye shielding and cannot resolve large impact parameters. In this case, a maximum impact parameter is set at the Debye length $b_{\max} = \lambda_D$. The limitation of the Lenard-Balescu approach is that it does not resolve large-angle scattering for very small impact parameters. In this case, a minimum impact parameter is set at $b_{\min}$, based on classical or quantum mechanical arguments (see equations 1.41 and 1.42). Both expansion procedures are based on $\ln\Lambda \gg 1$.

Convergent kinetic theories seek a unified approach that simultaneously captures both the large and small impact parameter limits. In some sense, this is not really necessary since each of the two

approaches resolves a different limit and thus provide a rigorous justification for both truncations of the b integral. However, it seems useful from a theoretical perspective to have such a unified theory, and it may be important to resolve each limit when considering higher order terms in the $\ln\Lambda$ expansion. Indeed, the major practical application motivating such research is to describe "moderately coupled" plasmas. Li and Petrasso [25] define a moderately coupled plasma as one in which $2 \lesssim \ln\Lambda \lesssim 10$. Strongly coupled plasmas are those with $\ln\Lambda \lesssim 2$ and weakly coupled plasmas have $\ln\Lambda \gtrsim 10$. In the previous theories, we have assumed weakly coupled plasma; but, as one transitions to moderately coupled plasmas higher order terms in the $\ln\Lambda$ expansion may need to be considered. Very high-density plasmas, such as those produced in laser-produced inertial confinement fusion, can have regions in which the plasma is moderately coupled (or, possibly, even strongly coupled).

With this motivation, Li and Petrasso [25] calculated the third order term in the Fokker-Planck collision operator

$$C_{\mathrm{FP}}(f_s, f_{s'}) = -\frac{\partial}{\partial \mathbf{v}} \cdot \sum_{s'} \Bigg\{ f_s(\mathbf{v}) \frac{\left\langle \Delta \mathbf{v} \right\rangle^{s/s'}}{\Delta t} - \frac{1}{2} \frac{\partial}{\partial \mathbf{v}} \cdot \left[f_s(\mathbf{v}) \frac{\left\langle \Delta \mathbf{v} \Delta \mathbf{v} \right\rangle^{s/s'}}{\Delta t} \right] + \frac{1}{6} \frac{\partial^2}{\partial \mathbf{v}\, \partial \mathbf{v}} \vdots \left[f_s(\mathbf{v}) \frac{\left\langle \Delta \mathbf{v}\, \Delta \mathbf{v}\, \Delta \mathbf{v} \right\rangle^{s/s'}}{\Delta t} \right] \Bigg\}. \tag{1.104}$$

They found the same equations for first

$$\frac{\langle \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} = \Gamma_{ss'} \frac{\partial H_{s'}(\mathbf{v})}{\partial \mathbf{v}} \tag{1.105}$$

and second

$$\frac{\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} = \Gamma_{ss'} \frac{\partial^2 G_{s'}(\mathbf{v})}{\partial \mathbf{v} \partial \mathbf{v}} - \frac{\Gamma_{ss'}}{\ln \Lambda_{ss'}} \left[\frac{3}{2} \frac{\partial^2 G_{s'}(\mathbf{v})}{\partial \mathbf{v} \partial \mathbf{v}} - \mathcal{I} H_{s'}(\mathbf{v}) \right] \tag{1.106}$$

order that were discussed in section 1.1.4 (except that they also included the $\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle_{\parallel} / \Delta t$ term that is typically neglected in the Fokker-Planck equation because it is higher order in $1/\ln\Lambda_{ss'}$). Here $H_{s'}$ and $G_{s'}$ are the Rosenbluth potentials from equations A.1 and A.2 . The new triplet order equation is the third rank tensor

$$\frac{\langle \Delta \mathbf{v} \Delta \mathbf{v} \Delta \mathbf{v} \rangle^{s/s'}}{\Delta t} = -\frac{1}{2} \frac{\Gamma_{ss'}}{\ln \Lambda_{ss'}} \frac{m_{ss'}}{m_s} \frac{\partial^2 \mathbf{\Phi}_{s'}(\mathbf{v})}{\partial \mathbf{v} \partial \mathbf{v}} \tag{1.107}$$

in which

$$\mathbf{\Phi}_{s'}(\mathbf{v}) \equiv \int d^3 v'\, \mathbf{u} |\mathbf{u}|\, f_{s'}(\mathbf{v}') \tag{1.108}$$

is a new vector potential that is analogous to the Rosenbluth potentials. Equation 1.107 shows that at third order in the Fokker-Planck expansion, there is no need to truncate any integrals. This is because $\Gamma_{ss'} \propto \ln \Lambda_{ss'}$ so equation 1.107 is independent of $\ln \Lambda_{ss'}$. The third (and higher) order in the expansion are thus dominated by large-angle (close interactions) rather than the typical small-angle Coulomb collisions that dominate at lower order; a fact that could be anticipated from the basic expansion technique. Thus, it seems that a convergent kinetic equation concerns only the typical low-order terms that are dominant in weakly-coupled plasma.

With the understanding that the truncation of b is only required at lowest order (for the terms kept in the conventional Landau and Lenard-Balescu equations), the contribution of a convergent kinetic theory appears to be a unified approach for determining the Coulomb logarithm. We will find that the collision terms describing wave-particle interactions from instabilities do not suffer from this logarithmic divergence issue. However, we wish to briefly mention previous work on convergent kinetic equations, because it is perhaps unsettling that traditional stable plasma kinetic theories do not self-consistently account for both limits of this integral (although, as we have discussed above, the kinetic theory does sit on a very firm foundation with rigorous approaches for determining each limit).

Hubbard [26], was the first to show that the small momentum transfer limit of the Boltzmann equation and the Lenard-Balescu equation could be combined to provide a convergent equation. This approach, in essence, adds the Boltzmann and Lenard-Balescu results, then subtracts the overlaying Landau equation: $C(f_s) = C_{\mathrm{B}}(f_s) + C_{\mathrm{LB}}(f_s) - C_{\mathrm{L}}(f_s)$, but does so early in the analysis so as to resolve the integrals. Similar approaches were pursued by Aono [27], Baldwin [28], Frieman and Book [29] and Gould and DeWitt [30], who used various methods including test-particle, BBGKY hierarchy, quantum mechanical and ladder diagram. A summary of these approaches, which shows the equivalence of the results, has been provided by Aono [31]. A more modern approach based on a quantum field theory derivation has also been developed by Brown, Preston and Singleton [32]. The results of the theories largely affirm the aforementioned truncation for the limits of b, especially for weakly coupled plasmas. An area of possible contention arises as one approaches a strongly coupled plasma. However, in this case the usual $1/\ln\Lambda$ expansion is no longer valid. One must use a fundamentally different approach to deal with strongly coupled plasmas that is not based on the large magnitude of $\Lambda \sim n\lambda_D^3$. It must also be quantum mechanical because the plasma density is necessarily large, and interactions close, in this

regime. Conventionally, part of the definition of "plasma" has been that there must be many particles in a Debye sphere: $4\pi n\lambda_D^3 \gg 1$ [33]. Thus, at least according to this conventional definition, strongly coupled plasmas are not plasmas at all. In the remainder of this work, we will be concerned only with conventional weakly coupled plasmas.

1.2 Previous Theories for Unstable Plasmas

Next, we turn to the topic that will be the focus of this work: scattering in unstable plasmas. The bulk of theory in this area is concentrated on turbulence, which assumes that fluctuation amplitudes are so large that they dominate scattering processes and also that they have ceased to grow due to nonlinear saturation mechanisms that arise when large amplitude fluctuations interact with one another. In turbulence theories, fluctuations with small wavenumber (long wavelength) are typically unstable and grow, but quickly break apart due to nonlinear interactions causing a cascade to larger and larger wavenumbers. At large enough wavenumbers, these fluctuations subsequently dissipate and transfer their energy back to the plasma. Such highly nonlinear states are what is typically studied because they are common in many astrophysics and fusion applications where strongly growing fluid instabilities are present. These instabilities are often absolute in the sense that they do not convect while they grow. Convective instabilities have a finite group velocity and thus propagate while they grow. Often these will convect out of the region of interest before reaching nonlinear amplitudes. Absolute instabilities, on the other hand, grow in time at each fixed spatial location and quickly reach large amplitudes where nonlinear saturation effects become important.

In this work we will be mainly concerned with fluctuations that do not interact with one another in a nonlinear way. This is not to say that the theory is linear, because the collision operator is a nonlinear expression in which instabilities can "feed back" to alter the "equilibrium" distribution function and change the instability growth rate. However, evolution of the equilibrium is assumed to happen over space and time scales much longer than those of fluctuations. Such approaches are often said to use a quasilinear approximation, although the term "quasilinear theory" is associated with the specific theory that we describe in the next section. Theories based on this new approach are best applied to plasmas with convective instabilities that either propagate out of the plasma, or region of interest, or modify

the equilibrium distribution to reduce the instability amplitude before nonlinear interactions between the fluctuations themselves become prominent. In this section we summarize previous theories in this area, which are commonly classified as theories of weakly unstable plasma.

1.2.1 Quasilinear Theory

The most prominent theory describing weakly unstable plasma is quasilinear theory. Quasilinear theory is considered a "collisionless" theory because it is based on the Vlasov equation [34]

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s} \mathbf{E} \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0, \tag{1.109}$$

which is the same as the kinetic equations from section 1.1, but with the collision operator set equal to zero. However, it is concerned with deriving an "effective" collision operator that describes scattering by wave-particle interactions in weakly unstable plasma.

Quasilinear theory was first developed by Vedenov, Velikhov and Sagdeev [8, 9] and independently by Drummond and Pines [10]. Other notable early references on quasilinear theory are Bernstein and Englemann [35] and Vedenov and Ryutov [36]. A detailed derivation of quasilinear theory is provided in section 3.1. The theory is based on separating f_s such that $f_s = f_{s,o} + f_{s,1}$ in which $f_{s,o}$ is essentially stationary on the shorter time and spatial scales of the fluctuating component $f_{s,1}$. It is assumed that only electrostatic fluctuations are present (although one can generalize the theory to include electromagnetic fluctuations). The result is the diffusion equation

$$\frac{\partial f_{s,o}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{s,o}}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{v}} \cdot \mathcal{D}_v \cdot \frac{\partial f_{s,o}}{\partial \mathbf{v}},$$

in which the velocity-space diffusion coefficient is

$$\mathcal{D}_v = \frac{q_s^2}{m_s^2} 8\pi \sum_j \int d^3k \frac{\mathbf{kk}}{k^4} \frac{\gamma_j \mathcal{E}_j^{\mathrm{ql}}(\mathbf{k})}{[(\omega_{R,j} - \mathbf{k} \cdot \mathbf{v})^2 + \gamma_j^2]}.$$

Here the subscript j represents the unstable modes and the spectral energy density is defined as

$$\mathcal{E}_j^{\mathrm{ql}}(\mathbf{k}) = \frac{|\hat{E}_1(\mathbf{k}, t=0)|^2}{(2\pi)^3 V} \frac{e^{2\gamma_j t}}{8\pi}.$$

We will study the quasilinear equation in detail in chapter 3. The salient features to notice here are that: (a) it is a diffusion equation; (b) it does not depend explicitly on each species in the plasma,

but only on the fluctuation spectrum (i.e. k-dependence) and its initial amplitude; and (c) the fluctuation source is taken as an input via the initial fluctuation amplitude $|\hat{E}_1(\mathbf{k}, t = 0)|^2$, which must be determined external to the theory. Fluctuations in conventional quasilinear theory can be from any electrostatic source, be it internally generated in the plasma, or from an externally applied wave. In developing a kinetic theory, we will be interested in instabilities that arise internal to a plasma.

1.2.2 Kinetic Theories With Instabilities

Kinetic theories of weakly unstable plasmas have been developed by Friemann and Rutherford [5] and Rogister and Oberman [6, 7]. Unlike quasilinear theory, these kinetic approaches are not based on the Vlasov equation. Instead they develop a collision operator that accounts for both particle-particle and wave-particle interactions. This basic idea is similar to what we will use in chapter 2 of this work to develop a collision operator, but our results differ substantially from previous theories due to specification of (or lack of specification of) the source of electrostatic fluctuations.

Rogister and Obermann [6, 7] used a discrete-particle approach, similar to what we will use in section 2.1, to derive a kinetic theory for weakly unstable plasmas. Their result is a collision operator that consists of a sum of terms. The part of this sum that describes particle-particle scattering is the Lenard-Balescu equation and the rest describes wave-particle interactions. The salient difference between Rogister and Obermann's result and what we derive in chapter 2.1 is that they do not specify a source of fluctuations, while we will associate the fluctuation source with discrete particle motion in the plasma. As a result, the Rogister-Obermann theory requires external specification of the fluctuation source; in a similar way to how quasilinear theory requires specification of $|E_1(\mathbf{k}, t = 0)|^2$. The initial fluctuation amplitude that must be specified in their theory comes about as $I_{\mathbf{k}}(0)$ in equation (22) of reference [6].

Friemann and Rutherford [5] used a BBGKY hierarchy method, similar to what we will use in section 2.2, to derive a kinetic theory for weakly unstable plasmas. They focused on nonlinear aspects such as mode coupling between unstable waves that enter the kinetic equation at higher order in the hierarchy expansion than we consider in this work. The part of their collision operator that described collisions between particles and collective fluctuations also depended on an initial fluctuation level that must be

determined external to the theory, just like the the Rogister-Oberman theory and quasilinear theory. Again, this is the main distinction between previous work and our approach.

1.2.3 Considerations of the Source of Fluctuations

Consideration of the discrete particle source of fluctuations in a plasma was first provided by Kent and Taylor [11] in 1969 (after the Rogister-Oberman, Friemann-Rutherford and quasilinear theories had been developed). Kent and Taylor used the WKB approximation to calculate the amplification of convective fluctuations from discrete particle motion. They focused on describing the fluctuation amplitude, rather than a kinetic equation for particle scattering, and emphasized drift-wave instabilities in magnetized inhomogeneous systems.

Baldwin and Callen derived a kinetic equation (collision operator) accounting for the discrete particle source of fluctuations and their effects on instability-enhanced collisional scattering for the specific case of loss-cone instabilities in magnetic mirror devices [12]. In the present work, we consider electrostatic instabilities in unmagnetized plasmas. However, the qualitative feature that the collision frequency due to instability-enhanced interactions scales as the product of $\delta/\ln\Lambda$ and the energy amplification due to fluctuations $[\exp(2\gamma t)]$ is common to both. Here δ is typically a small number $\sim 10^{-2} - 10^{-3}$, which depends on the fraction of wave-number space that is unstable. Although the Baldwin-Callen paper describes a specific example instability in magnetized plasmas, it has much in common with the present work because it developed a kinetic equation through a self-consistent treatment of fluctuations arising internal to the plasma from discrete particle motion.

1.3 Advantages of the Approach Taken in This Work

As section 1.2 mentions, the advantage of the kinetic theory developed in chapter 2 of this work is that it self-consistently accounts for a discrete particle source of fluctuations – which is the source whenever instabilities arise internal to a plasma. Sections 1.2.1 and 1.2.2 described previous kinetic and quasilinear theories where the collision operators required inputing the amplitude and spectrum of the fluctuation source ($|E_1(\mathbf{k}, t = 0)|^2$ in the quasilinear theory). These general formulations have the advantage that they can, in principle, accommodate whatever electrostatic fluctuation source one can

input; be it externally applied waves (e.g. from an antenna) or instabilities that are excited internal to the plasma. They have the disadvantage that the source fluctuation spectrum is often unknown. When these theories (especially quasilinear theory) are applied to situations where instabilities are excited internally, the source spectrum is often taken as a constant with an amplitude characteristic of the thermal fluctuation level. However, in these situations the source is due to discrete particles, and we will see in section 3.3 that the source fluctuation spectrum is significantly more complicated than is typically assumed (in particular it has a wave-number dependence that is determined by the instability). This issue is important for many applications to which quasilinear theory is applied. For example, the bump-on-tail instability is a textbook problem [37] where quasilinear theory is applied to an internally generated instability with a discrete particle source of fluctuations.

Aside from explicitly determining $|E_1(\mathbf{k}, t = 0)|^2$ for a discrete particle source, the collision operator we derive has other important distinguishing features. One of these is that it captures the effects of collisions in both stable and unstable plasmas (the kinetic theories of Rogister and Oberman [6] and Frieman and Rutherford [5] also do this; quasilinear theory does not). The result is a collision operator that consists of the sum of a stable plasma part (the Lenard-Balescu operator) and an instability-enhanced part (the new term, which we call the instability-enhanced operator). It can thus describe stable or unstable plasmas where one or the other term dominates, as well as marginally stable plasmas where the two terms can be comparable in magnitude.

Another distinguishing feature is that the resultant *total* collision operator (both the Lenard-Balescu and instability-enhanced terms) can be written as the sum of *component* collision operators for the interactions between individual species: $C(f_s) = \sum_{s'} C(f_s, f_{s'})$ in which s is the test species and s' are all the plasma species (including s itself). Neither quasilinear theory nor the previous kinetic theories have this feature; in section 3.5 we show that it requires specification of the discrete particle source of fluctuations. It is an important feature because in many applications one is interested not only in the total collisional interaction, but also in the collisional interaction between two (or more) particular species. For example, in the Langmuir's paradox problem we will be interested in electron-electron collisions and in the multi-ion-species Bohm problem in $s - s'$ collisions where each is an ion species with a different mass (or charge).

We will see in chapter 2 that the component collision operators have the Landau form with both

diffusion and drag components. In section 2.3, we show that the total collision operator has only a diffusion component because the sum of the drag terms over all s' species cancel out. However, the drag term is an important part of the component interaction, and this cannot be described by any previous theory. In section 3.4 we show that the ability to resolve component collisions leads to more restrictive conservation laws than quasilinear theory obeys, such as momentum lost by species s due to collisions with s' is gained by s'. It is also an important feature required to show that the unique equilibrium for collisions between any two species are Maxwellians with equal flow speeds and temperatures. This property will be essential in the Langmuir's paradox application. It is not, however, a property of the previous quasilinear or kinetic theories of unstable plasmas.

1.4 Application to Langmuir's Paradox

Langmuir's paradox is, perhaps, the oldest unsolved problem in plasma physics. In 1925, while developing the gas-filled incandescent lamp, Langmuir measured the electron distribution function in a 3 cm diameter discharge to be Maxwellian at all energies he could diagnose with an electrostatic probe (in excess of 50 eV) [13]. This was a surprising result because electrons with energy greater than the sheath potential drop, $e\Delta\phi_s \approx -T_e \ln\sqrt{2\pi m_e/M_i}$ ($\approx 5T_e$ for mercury), quickly escape the plasma and are lost to the boundary walls. Langmuir's experiment was a filament discharge creating a mercury plasma with electron (plasma) density $n_e \approx 10^{11}$ cm^{-3}, neutral density $\approx 10^{13}$ cm^{-3} (0.3 mTorr), and ion and electron temperatures of $T_i \approx 0.03$ eV and $T_e \approx 2$ eV respectively. For these parameters, the electron-electron collision length, using stable plasma theory, is approximately 30 cm which is much larger than the plasma length. Thus, one should expect the electron distribution to be essentially absent of particles beyond the 10 eV energy corresponding to the sheath. Langmuir also pointed out that he could attribute the vast majority of ionization events in the discharge to be due to the very same electrons on the Maxwellian tail (rather than the filament-emitted electrons, which energized his plasma) that should be missing according to the theory [13]. Since these electrons should rapidly escape, it was inexplicable how his discharge remained lit, and it suggested that some unknown mechanism for electron scattering was present. His measurements were named "Langmuir's paradox" by Gabor in 1955 [15], and they have remained a serious discrepancy in the kinetic theory of gas discharges.

In chapter 4, we consider details of the plasma-boundary transition in order to explain this paradox. This transition consists of the sheath potential drop, $\Delta\phi_s$, over a Debye length-scale $\lambda_{De} \equiv \sqrt{T_e/4\pi e n_e}$ region at the boundary surface, but also a much weaker presheath potential drop that extends further into the plasma. In the presheath, the electric potential typically drops $e\Delta\phi_{ps} \approx T_e/2$ over a distance characteristic of the ion-neutral collision mean free path $\lambda^{i/n} \gg \lambda_{De}$. The presheath was shown by Bohm [38] to be necessary in order to accelerate the ion fluid speed to a supersonic value $V_i \geq c_s \equiv \sqrt{T_e/M_i}$ at the sheath edge. We show that in Langmuir's discharge, ion-acoustic instabilities are present in the presheath which lead to an instability-enhanced collective response, and hence fluctuations, that cause electron-electron scattering to occur much more frequently than it does by Coulomb interactions alone. The calculation predicts an electron-electron collision length at least 100 times shorter than that calculated using stable plasma theory and the result is consistent with Langmuir's measurements. Our theory is well suited to this problem because the ion-acoustic instabilities are convective modes that travel through the presheath and are lost from the plasma while still in a linear growth regime.

Features of the kinetic theory that are essential for application to this problem are the ability to describe component interactions (electron-electron interactions in this case) and that Maxwellian is the unique equilibrium solution to the electron-electron component collision operator. These properties are both satisfied by the kinetic theory of chapter 2, but not by previous quasilinear or kinetic theories, which we will show in chapter 3.

1.5 Application to Determining the Bohm Criterion

A second outstanding problem that we apply our plasma kinetic theory to is determining the Bohm criterion for multiple-ion-species plasmas. This means determining the flow speed of each ion species, V_i, as it leaves a plasma and enters a sheath. Generalizing the conventional Bohm criterion to a plasma with multiple ion species (distinguished by different masses or charges) yields [39–41]

$$\sum_i \frac{n_{io}}{n_{eo}} \frac{c_{s,i}^2}{V_i^2} \leq 1. \tag{1.110}$$

Even when assuming equality holds, which is expected [42], equation 1.110 has an infinite number of possible solutions for more than one ion species. Finding the correct physical solution is what we mean

by determining the Bohm criterion. Previous theoretical work on this topic [43–47] predicts that the solution of equation 1.110 is that each ion species obtains its individual sound speed at the sheath edge: $V_i = c_{s,i} = \sqrt{T_e/M_i}$. However, experiments using laser-induced fluorescence have measured the speeds to be significantly different than this and much closer to another solution of equation 1.110, which is that each species obtains the same "system" sound speed

$$c_s \equiv \sqrt{\sum_i \frac{n_i}{n_e} c_{s,i}^2}$$

at the sheath edge [48–50]. Additional experimental evidence has been provided by ion-acoustic wave measurements [51, 52]. Oksuz *et al* [52] have measured that for two ion species plasmas the ion-acoustic wave speed at the sheath edge is typically twice what it is in the bulk plasma. Taking this observation as an ansatz, Lee *et al* [53] have shown that it implies each ion species enters the sheath at the common system sound speed. However, no physical mechanism has been suggested by which this solution is established.

In chapter 6, we show that when the presheath electric field drives the speeds of each ion species apart, due to their mass difference, a two-stream instability arises when their relative speed exceeds a critical value characteristic of their thermal speeds. As this occurs, a strong instability-enhanced collisional friction arises which pushes the speeds together. We calculate this instability-enhanced friction using our collision operator accounting for the two-stream instabilities. This shows that the two-stream instabilities create a very stiff system whereby if the relative flow between ion species exceeds the threshold value, the friction rapidly dominates the momentum balance and forces the speeds back to the critical relative flow. This provides a relation between the ion flow speeds, and thus determines which solution of equation 1.110 is obtained.

The expression we obtain for the critical relative flow speed depends on the relative densities of the ion species. It has the property that for very different densities, instabilities are not expected in the presheath. In this case, the difference in flow speeds is simply the difference in sound speeds of each species. For similar densities, however, the two-stream instability is strong when the difference in flow speeds exceeds a critical value that is on the order of the ion thermal speeds. In this case, the difference in flow speeds can be significantly smaller than it is when the density ratio of ion species is very large or small. Our theoretical predictions have been measured independently by Yip, Hershkowitz

and Severn [54] and they are in very good agreement with the experimentally measured values. We show this comparison in section 6.5.

The most important physical properties of the kinetic theory in this application are the ability to describe individual component collision operators and momentum conservation for collisions between individual species. Again, these properties are obeyed in the kinetic theory of chapter 2, but not by previous theories. It is also important that the fluctuation source be determined in order to calculate the expected collisional friction between species.

Chapter 2

Kinetic Theory of Weakly Unstable Plasma

This chapter provides two derivations of a plasma kinetic equation that includes the effects of conventional Coulomb collisions between particles as well as wave-particle collisions that arise from instabilities in a linear growth regime. The two derivations are based on fundamentally different approaches to describing the statistical evolution of a large number of interacting particles. The derivation in section 2.1 uses the "dressed test particle" approach. This is based on an appropriate ensemble average of the exact Klimontovich equation which describes the evolution of the 6-N dimensional distribution function of N particles in real and velocity phase-space. "Dressed" means that the Coulomb electric field of each particle is shielded due to polarization that is described by the plasma dielectric. The derivation in section 2.2 uses the statistical approach of the BBGKY hierarchy. This is based on building a hierarchy of equations from the exact Liouville equation which describes the evolution of the plasma described as a single system, or point, in a 6-N dimensional phase space. Each method leads to the same plasma kinetic equation, which includes the instability-enhanced collisional scattering.

2.1 Dressed Test Particle Approach

The dressed test particle approach, first developed by Dupree [55], starts by defining an exact distribution function for a particular species s as the sum over the location of each particle in a six-dimensional phase-space for velocity and position

$$F_s \equiv \sum_i^{N_s} \delta[\mathbf{x} - \mathbf{x}_i(t)]\, \delta[\mathbf{v} - \mathbf{v}_i(t)]. \tag{2.1}$$

Here $\mathbf{x}$ and $\mathbf{v}$ are the phase space coordinates while $\mathbf{x}_i$ and $\mathbf{v}_i$ represent the position of particle i (of species s) in phase space. F_s is a spiky function that is zero everywhere except where there is a particle. There are N_s of these spikes and in a typical plasma N_s is an extremely large number ($\sim 10^{10}$ for a low temperature laboratory plasma). Different species may be classified as particles with different charges and masses in the plasma. A simple example would be to classify electrons and protons as separate species in an electron-proton plasma.

2.1.1 Klimontovich Equation

An equation of motion for the distribution function F_s, called the Klimontovich equation [56], can be derived by taking a partial time derivative of F_s

$$\begin{aligned} \frac{\partial F_s}{\partial t} &= \frac{\partial}{\partial t}\sum_{i=1}^{N_s}\delta[\mathbf{x}-\mathbf{x}_i(t)]\delta[\mathbf{v}-\mathbf{v}_i(t)] \\ &= \sum_{i=1}^{N_s}\left(\frac{d\mathbf{x}_i}{dt}\cdot\frac{\partial}{\partial\mathbf{x}_i}+\frac{d\mathbf{v}_i}{dt}\cdot\frac{\partial}{\partial\mathbf{v}_i}\right)\delta[\mathbf{x}-\mathbf{x}_i(t)]\delta[\mathbf{v}-\mathbf{v}_i(t)]. \end{aligned} \tag{2.2}$$

Neglecting gravity, particles are influenced only by the total electric and magnetic fields at each location, so the free particle trajectories are given by the Lorentz force equation, $d\mathbf{x}_i/dt = \mathbf{v}_i$ and $d\mathbf{v}_i/dt = (q_i/m_i)\big[\mathbf{E}+(\mathbf{v}_i/c)\times\mathbf{B}\big]$. The electric and magnetic fields may consist of both fields produced by the charged particles in the plasma as well as externally applied fields; for example $\mathbf{E} = \mathbf{E}(\mathbf{x}_i,t)+\mathbf{E}_{\text{applied}}$. Under the assumption of only electric and magnetic forcing fields, equation 2.2 can be written

$$\frac{\partial F_s}{\partial t} = -\sum_{i=1}^{N_s}\left\{\mathbf{v}_i\cdot\frac{\partial}{\partial\mathbf{x}_i}-\frac{q_i}{m_i}\left[\mathbf{E}(\mathbf{x}_i,t)+\frac{\mathbf{v}_i}{c}\times\mathbf{B}(\mathbf{x}_i,t)\right]\cdot\frac{\partial}{\partial\mathbf{v}_i}\right\}\delta\big[\mathbf{x}-\mathbf{x}_i(t)\big]\delta\big[\mathbf{v}-\mathbf{v}_i(t)\big]. \tag{2.3}$$

Since the delta functions are zero everywhere except $\mathbf{x} = \mathbf{x}_i$ and $\mathbf{v} = \mathbf{v}_i$, we can rearrange this equation with $\mathbf{x}_i \leftrightarrow \mathbf{x}$ and $\mathbf{v}_i \leftrightarrow \mathbf{v}$. Also, we assume that particles with different charge and/or mass are classified as different species. With these, the Klimontovich equation for species s can be written

$$\frac{dF_s}{dt} = \frac{\partial F_s}{\partial t}+\mathbf{v}\cdot\frac{\partial F_s}{\partial\mathbf{x}}+\frac{q_s}{m_s}\big(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\big)\cdot\frac{\partial F_s}{\partial\mathbf{v}} = 0 \tag{2.4}$$

in which $\mathbf{E} = \mathbf{E}(\mathbf{x},t)$, $\mathbf{B} = \mathbf{B}(\mathbf{x},t)$. The quantity d/dt is the convective derivative in the six-dimensional phase space $(\mathbf{x},\mathbf{v})$. The fact that $dF_s/dt = 0$ shows that along the free particle trajectories (characteristics) F_s is constant.

2.1.2 Plasma Kinetic Equation

The plasma kinetic equation can be derived from the Klimontovich equation 2.4 by an appropriate average of F_s that separates the ensemble averaged and discrete particle components of F_s, $F_s = f_s + \delta f_s$ where $f_s \equiv \langle F_s \rangle$ and $\langle \delta f_s \rangle = 0$. Here, the bracket denotes an ensemble average. Analogous notation is used for the electric and magnetic fields, e.g., $\mathbf{E} \to \mathbf{E} + \delta\mathbf{E}$. The desired plasma kinetic equation is obtained by putting these definitions into the Klimontovich equation, then ensemble averaging the result. We will then use a linear closure scheme to determine the particle-discreteness distribution δf_s [22, 33].

Ensemble averaging the Klimontovich equation yields the plasma kinetic equation

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\frac{\partial f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s}\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\frac{\partial f_s}{\partial \mathbf{v}} = -\frac{q_s}{m_s}\left\langle\left(\delta\mathbf{E} + \frac{\mathbf{v}}{c}\times\delta\mathbf{B}\right)\cdot\frac{\partial \delta f_s}{\partial \mathbf{v}}\right\rangle = C(f_s) \tag{2.5}$$

in which $C(f_s)$ is the total collision operator. The collision operator can be written in terms of the collisional current $\mathbf{J}_v$

$$C(f_s) = -\frac{\partial}{\partial \mathbf{v}}\cdot\mathbf{J}_v \quad \text{where} \quad \mathbf{J}_v \equiv \frac{q_s}{m_s}\left\langle\left(\delta\mathbf{E} + \frac{\mathbf{v}}{c}\times\delta\mathbf{B}\right)\delta f_s\right\rangle. \tag{2.6}$$

In equation 2.5 we have used the notation $\langle\mathbf{E}\rangle = \mathbf{E}$ and $\langle\mathbf{B}\rangle = \mathbf{B}$.

Subtracting the plasma kinetic equation 2.5 from the Klimontovich equation 2.4 gives a kinetic equation for the perturbed distribution function

$$\underbrace{\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{q_s}{m_s}\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\frac{\partial}{\partial \mathbf{v}}\right]}_{\text{Vlasov operator}}\delta f_s = \underbrace{-\frac{q_s}{m_s}\left(\delta\mathbf{E} + \frac{\mathbf{v}}{c}\times\delta\mathbf{B}\right)\cdot\frac{\partial f_s}{\partial \mathbf{v}}}_{\text{linear driving term}} \tag{2.7}$$

$$+ \underbrace{\frac{q_s}{m_s}\left[\left\langle\left(\delta\mathbf{E} + \frac{\mathbf{v}}{c}\times\delta\mathbf{B}\right)\cdot\frac{\partial \delta f_s}{\partial \mathbf{v}}\right\rangle - \left(\delta\mathbf{E} + \frac{\mathbf{v}}{c}\times\delta\mathbf{B}\right)\cdot\frac{\partial \delta f_s}{\partial \mathbf{v}}\right]}_{\text{nonlinear driving term}}.$$

Equation 2.7, along with Maxwell's equations, provides a closed system that exactly determines the collision operator. However, in practice this would be extremely difficult to solve because equation 2.7 is a very complicated nonlinear equation.

The way we proceed is to neglect the nonlinear terms on the right side of equation 2.7 under the assumption that $\mathcal{O}(\delta)$ terms are much smaller than the ensemble averaged quantities: thus $\delta\mathbf{E} f_s \gg \delta\mathbf{E}\delta f_s$, etc. This leaves

$$\frac{\partial \delta f_s}{\partial t} + \mathbf{v}\cdot\frac{\partial \delta f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s}\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\frac{\partial \delta f_s}{\partial \mathbf{v}} = -\frac{q_s}{m_s}\left(\delta\mathbf{E} + \frac{\mathbf{v}}{c}\times\delta\mathbf{B}\right)\cdot\frac{\partial f_s}{\partial \mathbf{v}}. \tag{2.8}$$

In a stable plasma, it can be shown that $\delta f_s/f_s \sim \mathcal{O}(\Lambda^{-1}) \ll 1$ where $\Lambda \sim n\lambda_D^3 \gg 1$ is the plasma parameter [57]. Thus, for a stable plasma, neglecting the nonlinear terms is an excellent approximation. Here we will be interested in unstable plasmas, and will find that in this case the small parameter is multiplied by a factor characteristic of the amplification of collisional scattering due to instabilities. Thus, the strength of this amplification factor will ultimately determine the limitation of our kinetic theory. We refer to such large instability amplitudes as nonlinear because they imply that the nonlinear terms, of $\mathcal{O}(\delta^2)$, are at least comparable in magnitude to the linear terms, of $\mathcal{O}(\delta)$, in equation 2.7. When this happens, nonlinear wave saturation mechanisms are expected to become important.

Even with the linearized approximation, solving equation 2.8 along with Maxwell's equations for the collision operator of equation 2.6 presents a very complicated problem. In this work we will only be interested in electrostatic instabilities. Thus, we take $\delta\mathbf{B} = 0$ and the only relevant Maxwell equation becomes Gauss law; this elimination of electromagnetic instabilities provides a considerable simplification. Aside from in appendix B, we also assume that there is no "ensemble averaged," i.e., equilibrium, electric or magnetic fields, $\langle\mathbf{E}\rangle = 0$ and $\langle\mathbf{B}\rangle = 0$. This is also assumed in Lenard-Balescu theory. However, plasmas often do generate equilibrium fields through currents and self-polarization in the plasma, as well as from externally applied fields. In fact, in the applications portion of this work (chapters 4, 5 and 6) weak equilibrium fields will be expected. Appendix B provides derivations for collision operators that include the effects of equilibrium electric and magnetic fields (these still assume electrostatic fluctuations). The results of appendix B show that equilibrium electric fields modify the collision operator when the gradient scale length of the potential is at least as short as k^{-1} where k^{-1} is the relevant unstable wavelength (for unstable plasmas) or the Debye length (for stable plasmas). For equilibrium magnetic fields, modifications to the analysis of this chapter occur when the gyroradius is comparable to, or smaller than, k^{-1}. The weak fields present in the applications we are interested in in this dissertation result in negligible modifications to the collision operator derived in this chapter; see appendices B.2 and B.3 for details. For other applications, particularly where strong magnetic fields are present, the collision operator may be modified and the methods of appendix B may be useful. The theory of how equilibrium fields modify collision operators is a largely unexplored area of plasma kinetic theory.

Applying the aforementioned assumptions to equation 2.5, the plasma kinetic equation for electrostatic fluctuations in equilibrium field-free plasma can then be written

$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} = C(f_s) \tag{2.9}$$

where the collision operator and collisional current are now

$$C(f_s) \equiv -\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{J}_v, \quad \text{and} \quad \mathbf{J}_v \equiv \frac{q_s}{m_s} \langle \delta\mathbf{E}\, \delta f_s \rangle. \tag{2.10}$$

Equation 2.7 can now be written

$$\frac{\partial \delta f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial \delta f_s}{\partial \mathbf{x}} = -\frac{q_s}{m_s} \delta\mathbf{E} \cdot \frac{\partial f_s}{\partial \mathbf{v}}. \tag{2.11}$$

In section 2.1.3 we will use equation 2.11 along with Gauss's law,

$$\frac{\partial}{\partial \mathbf{x}} \cdot \delta\mathbf{E} = 4\pi \sum_s q_s \int d^3v\, \delta f_s, \tag{2.12}$$

to derive a collision operator, $C(f_s)$, for plasmas that are either stable or unstable in a finite space-time domain. This approach is formally valid as long as

$$\left| \delta\mathbf{E} \cdot \frac{\partial \delta f_s}{\partial \mathbf{v}} - \left\langle \delta\mathbf{E} \cdot \frac{\partial \delta f_s}{\partial \mathbf{v}} \right\rangle \right| \ll \left| \delta\mathbf{E} \cdot \frac{\partial f_s}{\partial \mathbf{v}} \right|. \tag{2.13}$$

In section 2.5 we show that for $\omega \ll k v_{Te}$ this is equivalent to $q\delta\phi/T_e \lesssim 1$. Absolute instabilities must be confined to a finite time domain and convective instabilities to a finite space domain. If instabilities are allowed to grow over a long enough domain to violate equation 2.13, then nonlinear or turbulence methods must be used [58].

2.1.3 Collision Operator Derivation

To solve for the collision operator, we apply a combined Fourier transform in space and Laplace transform in time according to the definitions [for an arbitrary function $g(\mathbf{x}, t)$]

$$\mathcal{FL}\{g(\mathbf{x},t)\} = \hat{g}(\mathbf{k},\omega) = \int d^3x \int_0^\infty dt e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} g(\mathbf{x},t), \tag{2.14}$$

with the inverse given by

$$(\mathcal{FL})^{-1}\{\hat{g}(\mathbf{k},\omega)\} = g(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{d\omega}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \hat{g}(\mathbf{k},\omega). \tag{2.15}$$

We assume that the equilibrium f_s evolves on much longer space and timescales $(\bar{\mathbf{x}}, \bar{t})$ than δf_s; thus, f_s is independent of the short space and time scales $(\mathbf{x}, t)$ of the transform defined in equations 2.14 and 2.15. Applying this combined transform to equation 2.11 yields

$$\delta \hat{f}_s(\mathbf{k}, \mathbf{v}, \omega) = \frac{i\, \delta \tilde{f}_s(\mathbf{k}, \mathbf{v}, t' = 0)}{\omega - \mathbf{k} \cdot \mathbf{v}} - \frac{q_s}{m_s} \mathbf{k} \cdot \frac{\partial f_s}{\partial \mathbf{v}} \frac{\delta \hat{\phi}}{\omega - \mathbf{k} \cdot \mathbf{v}}, \tag{2.16}$$

where the "hat" denotes Fourier and Laplace transformed variables and the "tilde" denotes only Fourier transformed variables. Here $\delta f_s(t = 0)$ is the initial condition determined from the "exact" distribution $\delta f_s = F_s - f_s$. We have also written $\delta \mathbf{E}$ in terms of the electric potential (since we assume only electrostatic fluctuations are present): $\delta \mathbf{E}(\mathbf{x}, t) = -\partial \delta \phi(\mathbf{x}, t)/\partial \mathbf{x}$.

Substituting equation 2.16 into the Fourier-Laplace transform of Gauss's law, equation 2.12, leads to

$$\delta \hat{\phi}(\mathbf{k}, \omega) = \sum_s \frac{4\pi q_s}{k^2 \hat{\varepsilon}(\mathbf{k}, \omega)} \int d^3v \frac{i\, \delta \tilde{f}_s(t = 0)}{\omega - \mathbf{k} \cdot \mathbf{v}}, \tag{2.17}$$

where

$$\boxed{\hat{\varepsilon}(\mathbf{k}, \omega) = 1 + \sum_s \frac{4\pi q_s^2}{k^2 m_s} \int d^3v \frac{\mathbf{k} \cdot \partial f_s / \partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}}} \tag{2.18}$$

is the dielectric function for electrostatic fluctuations in equilibrium-field-free plasma. Equation 2.17 can be simplified by substituting in the combined transform of $\delta f_s(t = 0) = F_s(t = 0) - f_s$, which is

$$\delta \tilde{f}_s(t = 0) = \sum_{i=1}^{N} e^{-i\mathbf{k} \cdot \mathbf{x}_{io}} \delta(\mathbf{v} - \mathbf{v}_{io}) - (2\pi)^3 \delta(\mathbf{k}) f_s, \tag{2.19}$$

where $\mathbf{v}_{io} \equiv \mathbf{v}_i(t = 0)$, to give

$$\delta \hat{\phi}(\mathbf{k}, \omega) = \sum_{j=1}^{N} \frac{4\pi q_j}{k^2 \hat{\varepsilon}(\mathbf{k}, \omega)} \frac{i\, e^{-i\mathbf{k} \cdot \mathbf{x}_{jo}}}{\omega - \mathbf{k} \cdot \mathbf{v}_{jo}}. \tag{2.20}$$

Here we have used for the initial conditions that F_s satisfies $F_s(t = 0) = \sum_j \delta(\mathbf{x} - \mathbf{x}_{jo}) \delta(\mathbf{v} - \mathbf{v}_{jo})$ and applied the assumption that f_s is essentially uniform in space relative to spatial scales of δf_s, which implies that the Fourier terms of f_s are given by $(2\pi)^3 \delta(\mathbf{k}) f_s$. The term involving f_s in equation 2.19 produces no contribution to $\delta \phi$ because of quasineutrality:

$$\sum_s q_s \int d^3v \frac{\delta(\mathbf{k}) f_s}{\omega - \mathbf{k} \cdot \mathbf{v}} = \frac{\delta(\mathbf{k})}{\omega} \sum_s n_s q_s = 0. \tag{2.21}$$

Using equations 2.19 and 2.20, we find an expression for $\delta \hat{f}_s(\mathbf{k}, \mathbf{v}, \omega)$ from equation 2.16:

$$\delta \hat{f}_s = -\frac{i(2\pi)^3 \delta(\mathbf{k}) f_s}{\omega - \mathbf{k} \cdot \mathbf{v}} + \sum_{i=1}^{N} \left[\frac{i e^{-i\mathbf{k} \cdot \mathbf{x}_{io}} \delta(\mathbf{v} - \mathbf{v}_{io})}{\omega - \mathbf{k} \cdot \mathbf{v}} - \frac{4\pi q_s q_i}{m_s k^2 \hat{\varepsilon}(\mathbf{k}, \omega)} \frac{i\, \mathbf{k} \cdot \partial f_s / \partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} \frac{e^{-i\mathbf{k} \cdot \mathbf{x}_{io}}}{\omega - \mathbf{k} \cdot \mathbf{v}_{io}} \right], \tag{2.22}$$

which along with equation 2.20 determines the transform of the collision operator.

Since the transform of the collisional current, $\hat{\mathbf{J}}_v$, is the ensemble average of the convolution of electric field and distribution perturbations, it is convenient to define different transform variables for $\delta\hat{\mathbf{E}}$ and $\delta\hat{f}_s$. Keeping the notation of equation 2.22 the same and changing that of equation 2.20, we write

$$\delta\hat{\mathbf{E}}(\mathbf{k}',\omega') = \sum_{j=1}^{N} \frac{4\pi q_j}{k'^2\hat{\varepsilon}(\mathbf{k}',\omega')} \frac{\mathbf{k}' e^{-i\mathbf{k}'\cdot\mathbf{x}_{jo}}}{\omega' - \mathbf{k}'\cdot\mathbf{v}_{jo}}. \tag{2.23}$$

Then the transformed collisional current is defined by

$$\hat{\mathbf{J}}_v(\mathbf{k},\mathbf{k}',\mathbf{v},\omega,\omega') = \frac{q_s}{m_s}\left\langle \delta\hat{\mathbf{E}}(\omega',\mathbf{k}')\, \delta\hat{f}_s(\omega,\mathbf{k},\mathbf{v})\right\rangle, \tag{2.24}$$

where the ensemble average is [22]

$$\langle \ldots \rangle \equiv \prod_{l=1}^{N} \int d^3x_{lo} d^3v_{lo} \frac{f_l(\mathbf{v}_{lo})}{(nV)^N}(\ldots), \tag{2.25}$$

in which n denotes density and V denotes volume.

Taking the ensemble average of the product of equations 2.22 and 2.23 gives an array of terms:

$$\hat{\mathbf{J}}_v = \frac{q_s}{m_s} \prod_{l=1}^{N} \int d\Gamma_l \underbrace{\left[\sum_{j=1}^{N} \frac{4\pi q_j}{k'^2\hat{\varepsilon}(\mathbf{k}',\omega')} \frac{\mathbf{k}' e^{-i\mathbf{k}'\cdot\mathbf{x}_{jo}}}{\omega' - \mathbf{k}'\cdot\mathbf{v}_{jo}}\right]}_{\mathbf{j}} \times \tag{2.26}$$

$$\times \left\{ \underbrace{\sum_{i=1}^{N} \frac{ie^{-i\mathbf{k}\cdot\mathbf{x}_{io}}}{\omega - \mathbf{k}\cdot\mathbf{v}} \left[\delta(\mathbf{v} - \mathbf{v}_{io}) - \frac{4\pi q_s q_i}{m_s k^2 \hat{\varepsilon}(\mathbf{k},\omega)} \frac{i\mathbf{k}\partial f_s/\partial \mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}_{io}}\right]}_{\boxed{\text{i}}} - \frac{i(2\pi)^3 f_s(\mathbf{v})\delta(\mathbf{k})}{\omega - \mathbf{k}\cdot\mathbf{v}} \right\}$$

in which

$$d\Gamma_l \equiv \frac{d^3x_{lo} d^3v_{lo}}{(nV)^N} f_l(\mathbf{v}_{lo}). \tag{2.27}$$

This array can be written term-by-term as

$$\begin{aligned} \hat{\mathbf{J}}_v = \frac{q_s}{m_s} \int d\Gamma_{\mathbf{1}} \int d\Gamma_{\mathbf{2}} \ldots \int d\Gamma_{\mathbf{N}} \Big\{ & \quad \mathbf{1} \times \boxed{1} + \mathbf{1} \times \boxed{2} + \ldots + \mathbf{1} \times \boxed{\text{N}} \\ & + \mathbf{2} \times \boxed{1} + \mathbf{2} \times \boxed{2} + \ldots + \mathbf{2} \times \boxed{\text{N}} \\ & \qquad \vdots \\ & + \mathbf{N} \times \boxed{1} + \mathbf{N} \times \boxed{2} + \ldots + \mathbf{N} \times \boxed{\text{N}} \\ & + [\mathbf{1} + \mathbf{2} + \ldots + \mathbf{N}] \Big[-i(2\pi)^3 \delta(\mathbf{k}) f_s(\mathbf{v})/(\omega - \mathbf{k}\cdot\mathbf{v}) \Big] \Big\}. \end{aligned} \tag{2.28}$$

For unlike particle terms, $i \neq \boxed{\text{j}}$ (the off-diagonal terms in equation 2.28), the $\mathbf{x}_{lo}$ integral yields $(2\pi)^3\delta(\mathbf{k}')$. Since the rest of these terms tend to zero in the limit that $\mathbf{k}' \to 0$, the "unlike" particle terms vanish upon inverse Fourier transforming. This can be shown explicitly by first inverting the Laplace transforms, then using the definition of $\hat{\varepsilon}$ from equation 2.18. The $\mathbf{k}'$ integral becomes $h(\mathbf{k}')\mathbf{k}'\delta(\mathbf{k}')$, where $h(\mathbf{k}') \to c$ in which c is a constant as $\mathbf{k}' \to 0$. Thus, these terms are zero upon integrating over $\mathbf{k}'$. For the same reason, the terms in the bottom row of equation 2.28 vanish as well. We are then left with only "like" particle correlations ($i = \boxed{\text{j}}$) after the ensemble average. After the trivial $N-1$ integrals where $i \neq l$, we are left with

$$\mathbf{J}_v = \frac{4\pi i\, q_s \mathbf{k}'}{m_s\, k'^2 \hat{\varepsilon}(\mathbf{k}',\omega')\,(\omega - \mathbf{k}\cdot\mathbf{v})} \sum_{i=1}^{N} \frac{q_i}{nV} \int d^3v_{io} \frac{f_{io}(\mathbf{v}_{io})}{\omega' - \mathbf{k}'\cdot\mathbf{v}_{io}} \delta(\mathbf{v} - \mathbf{v}_{io}) \int d^3x_{io} e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}_{io}} \tag{2.29}$$
$$- \frac{(4\pi)^2 q_s^2\, i\mathbf{k}'\mathbf{k}\cdot\partial f_s/\partial\mathbf{v}}{m_s^2\, k^2 k'^2\, \hat{\varepsilon}(\mathbf{k}',\omega')\hat{\varepsilon}(\mathbf{k},\omega)(\omega - \mathbf{k}\cdot\mathbf{v})} \sum_{i=1}^{N} \frac{q_i^2}{nV} \int d^3v_{io} \frac{f_{io}(\mathbf{v}_{io})}{(\omega' - \mathbf{k}'\cdot\mathbf{v}_{io})(\omega - \mathbf{k}\cdot\mathbf{v}_{io})} \int d^3x_{io} e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}_{io}}$$

The $\mathbf{x}_{io}$ integrals in equation 2.29 are

$$\int d^3x_{io} \exp\big[-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}_{io}\big] = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}'), \tag{2.30}$$

and $\mathbf{v}_{io}$ is a dummy variable of integration. The sum over all particles becomes simply the total number of particles in the volume, $\sum_{i=1}^{N}/V = N/V = n$. Labeling $\mathbf{v}_{io} = \mathbf{v}'$ and $\sum_l q_l^2 f_l = \sum_{s'} q_{s'}^2 f_{s'}$, and noting that the terms with $\delta(\mathbf{v} - \mathbf{v}_{lo})$ obey $\int d^3v_{lo} f_l(\mathbf{v}_{lo})\delta(\mathbf{v} - \mathbf{v}_{lo}) = \int d^3v'\, f_s(\mathbf{v}')\delta(\mathbf{v} - \mathbf{v}')$ (since the $\mathbf{v}$ is associated with species s not s'), the transformed collisional current is

$$\hat{\mathbf{J}}_v = \frac{4\pi q_s^2}{m_s k^2} \int d^3v' \frac{i(2\pi)^3\mathbf{k}'\delta(\mathbf{k}+\mathbf{k}')}{\hat{\varepsilon}(\mathbf{k}',\omega')(\omega' - \mathbf{k}'\cdot\mathbf{v}')} \left[\frac{f_s(\mathbf{v}')\,\delta(\mathbf{v} - \mathbf{v}')}{\omega - \mathbf{k}\cdot\mathbf{v}} - \sum_{s'} \frac{4\pi q_{s'}^2}{k^2 m_s} \frac{f_{s'}(\mathbf{v}')\,\mathbf{k}\cdot\partial f_s/\partial\mathbf{v}}{(\omega - \mathbf{k}\cdot\mathbf{v})(\omega - \mathbf{k}\cdot\mathbf{v}')\hat{\varepsilon}(\mathbf{k},\omega)}\right]. \tag{2.31}$$

Symmetry between the two terms in this expression becomes explicit by evaluating the trivial $\mathbf{v}'$ integral in the first term, then multiplying this term by $\hat{\varepsilon}/\hat{\varepsilon}$ where the numerator is written in terms of equation 2.18, but with the substitutions of the dummy variables $\mathbf{v} \leftrightarrow \mathbf{v}'$ and $s \leftrightarrow s'$. This yields

$$\hat{\mathbf{J}}_v = \sum_{s'} \frac{(4\pi)^2 q_{s'}^2 q_s^2}{m_s k^4} \int d^3v' \frac{i(2\pi)^3\mathbf{k}'\delta(\mathbf{k}+\mathbf{k}')}{\hat{\varepsilon}(\mathbf{k}',\omega')\hat{\varepsilon}(\mathbf{k},\omega)(\omega - \mathbf{k}\cdot\mathbf{v})(\omega - \mathbf{k}\cdot\mathbf{v}')} \tag{2.32}$$
$$\times \left[\frac{f_s(\mathbf{v})\,\mathbf{k}\cdot\partial f_{s'}(\mathbf{v}')/\partial\mathbf{v}'}{m_{s'}(\omega' - \mathbf{k}'\cdot\mathbf{v})} - \frac{f_{s'}(\mathbf{v}')\,\mathbf{k}\cdot\partial f_s(\mathbf{v})/\partial\mathbf{v}}{m_s(\omega' - \mathbf{k}'\cdot\mathbf{v}')}\right]$$
$$+ \frac{4\pi q_s^2}{m_s k^2} \frac{i(2\pi)^3\mathbf{k}'\delta(\mathbf{k}+\mathbf{k}')f_s(\mathbf{v})}{\hat{\varepsilon}(\mathbf{k}',\omega')(\omega' - \mathbf{k}'\cdot\mathbf{v})(\omega - \mathbf{k}\cdot\mathbf{v})\hat{\varepsilon}(\mathbf{k},\omega)}.$$

The last term in equation 2.32 vanishes upon inverse Fourier transforming because it has odd parity in $\mathbf{k}$ after the $\mathbf{k}'$ integral.

Next, multiplying the first term in equation 2.32 by $(\omega' - \mathbf{k}' \cdot \mathbf{v}')/(\omega' - \mathbf{k}' \cdot \mathbf{v}')$, the term with $\mathbf{k}' \cdot \mathbf{v}'$ in the numerator will vanish upon performing the d^3k integrals because it is an odd function of $\mathbf{k}$. Similarly, for the second term we multiply by $(\omega' - \mathbf{k}' \cdot \mathbf{v})/(\omega' - \mathbf{k}' \cdot \mathbf{v})$, and the $\mathbf{k}' \cdot \mathbf{v}$ term vanishes. Rearraning the result, we find that we can write the collisional current in the "Landau" form [1]

$$\mathbf{J}_v = \int d^3v' \, \mathcal{Q}(\mathbf{v}, \mathbf{v}') \cdot \left(\frac{1}{m_{s'}} \frac{\partial}{\partial \mathbf{v}'} - \frac{1}{m_s} \frac{\partial}{\partial \mathbf{v}} \right) f_s(\mathbf{v}) f_{s'}(\mathbf{v}') \tag{2.33}$$

where $\mathcal{Q}$ is the tensor kernel

$$\mathcal{Q}(\mathbf{v}, \mathbf{v}') \equiv \frac{(4\pi)^2 q_s^2 q_{s'}^2}{m_s} \int \frac{d^3k}{(2\pi)^3} \frac{-i\,\mathbf{k}\mathbf{k}}{k^4} p_1(\mathbf{k}) p_2(\mathbf{k}), \tag{2.34}$$

in which p_1 and p_2 are defined by

$$p_1(\mathbf{k}) \equiv \int_{-\infty + i\sigma}^{\infty + i\sigma} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\hat{\varepsilon}(\mathbf{k}, \omega)(\omega - \mathbf{k} \cdot \mathbf{v})(\omega - \mathbf{k} \cdot \mathbf{v}')} \tag{2.35}$$

and

$$p_2(\mathbf{k}) \equiv \int_{-\infty + i\sigma}^{\infty + i\sigma} \frac{d\omega'}{2\pi} \frac{\omega' e^{-i\omega' t}}{\hat{\varepsilon}(-\mathbf{k}, \omega')(\omega' + \mathbf{k} \cdot \mathbf{v})(\omega' + \mathbf{k} \cdot \mathbf{v}')}. \tag{2.36}$$

Writing $\mathbf{J}_v$ in the Landau form of equation 2.33 will be convenient for illuminating the physics embedded in the collision operator as well as for proving important physical properties of the collision operator in section 3.4.

The integrals in p_1 and p_2 can be evaluated along the Landau contour using Cauchy's integral theorem to give

$$p_1 = i \left[\sum_j \frac{e^{-i\omega_j t}}{\partial \hat{\varepsilon}(\mathbf{k}, \omega)/\partial \omega|_{\omega_j} (\omega_j - \mathbf{k} \cdot \mathbf{v})(\omega_j - \mathbf{k} \cdot \mathbf{v}')} - \frac{i\pi \delta[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')]}{\hat{\varepsilon}(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})} e^{-i\mathbf{k} \cdot \mathbf{v}' t} \right] \tag{2.37}$$

where j denotes each mode, i.e., the dispersion relations, which are the roots of the dielectric function $\hat{\varepsilon}(\mathbf{k}, \omega_j) = 0$ from equation 2.18. In equation 2.37 we have combined the last two terms which come from the inverse Laplace transform by using the fact that $\exp\left[-i\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')t\right]$ is rapidly oscillating for large t, except at $\mathbf{v} = \mathbf{v}'$, so $\hat{\varepsilon}(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}') \approx \hat{\varepsilon}(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})$. Furthermore, we have identified the relation

$$-\frac{e^{-i\mathbf{k} \cdot \mathbf{v}' t}}{\hat{\varepsilon}(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}')} \left(\frac{1 - e^{-i\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')t}}{\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')} \right) \approx -\frac{i\pi \delta[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')]}{\hat{\varepsilon}(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})} e^{-i\mathbf{k} \cdot \mathbf{v}' t} \tag{2.38}$$

where the Dirac delta function definition is strictly correct only in the limit $t \to \infty$. However, it is a good approximation here because the timescale for variations in f_s, is much longer than the timescale for fluctuations [$(\mathbf{k}\cdot\mathbf{v})^{-1}$ here]. By similar arguments as used in 2.37, equation 2.36 becomes

$$p_2 = i\left[\sum_j \frac{\omega_j'\ e^{-i\omega_j' t}}{\frac{\partial\hat{\varepsilon}(-\mathbf{k},\omega')}{\partial\omega'}\big|_{\omega_j'}(\omega_j'+\mathbf{k}\cdot\mathbf{v})(\omega_j'+\mathbf{k}\cdot\mathbf{v}')} + \frac{e^{i\mathbf{k}\cdot\mathbf{v}t}}{\hat{\varepsilon}(-\mathbf{k},-\mathbf{k}\cdot\mathbf{v})} + \frac{i\pi\mathbf{k}\cdot\mathbf{v}'\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]e^{i\mathbf{k}\cdot\mathbf{v}t}}{\hat{\varepsilon}(-\mathbf{k},-\mathbf{k}\cdot\mathbf{v})}\right] \quad (2.39)$$

in which ω_j' solves $\hat{\varepsilon}(-\mathbf{k},\omega')=0$.

Putting the product of equations 2.37 and 2.39 into equation 2.34 gives an integral expression with six terms in the integrand. One term, which is the product of the last terms from equations 2.37 and 2.39, is an odd function of $\mathbf{k}$ and therefore vanishes after integration. Three of the terms are rapidly oscillating in time $\sim \exp(\pm i\mathbf{k}\cdot\mathbf{v}t)$ and provide negligible contributions after integration compared to the remaining two terms which survive. We are then left with the following expression:

$$\mathcal{Q} = \frac{2q_s^2q_{s'}^2}{m_s}\int d^3k\,\frac{\mathbf{kk}}{k^4}\left[\frac{\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]}{\hat{\varepsilon}(\mathbf{k},\mathbf{k}\cdot\mathbf{v})\hat{\varepsilon}(-\mathbf{k},-\mathbf{k}\cdot\mathbf{v})}\right. \quad (2.40)$$
$$\left. + \frac{i}{\pi}\sum_j \frac{\omega_j' e^{-i\omega_j' t}}{\partial\hat{\varepsilon}(-\mathbf{k},\omega')/\partial\omega'\big|_{\omega_j'}(\omega_j'+\mathbf{k}\cdot\mathbf{v})(\omega_j'+\mathbf{k}\cdot\mathbf{v}')}\ \frac{e^{-i\omega_j t}}{\partial\hat{\varepsilon}(\mathbf{k},\omega)/\partial\omega\big|_{\omega_j}(\omega_j-\mathbf{k}\cdot\mathbf{v})(\omega_j-\mathbf{k}\cdot\mathbf{v}')}\right].$$

Equation 2.40 can be further simplified by applying the reality conditions: $\hat{\varepsilon}(-\mathbf{k},-\mathbf{k}\cdot\mathbf{v}) = \hat{\varepsilon}^*(\mathbf{k},\mathbf{k}\cdot\mathbf{v})$, where $*$ denotes the complex conjugate, and $\omega_j = \omega_{R,j} + i\gamma_j$ (where $\omega_{R,j}$ and γ_j are the real and imaginary parts of the j^{th} solution of the dispersion relation) obey the properties that $\omega_{R,j}$ is an odd function of $\mathbf{k}$ while γ_j is an even function of $\mathbf{k}$. It follows then that $\omega_j' = -\omega_j^*$, and

$$\left.\frac{\partial\hat{\varepsilon}(-\mathbf{k},\omega')}{\partial\omega'}\right|_{\omega_j'} = -\left.\frac{\partial\hat{\varepsilon}^*(\mathbf{k},\omega)}{\partial\omega}\right|_{\omega_j}. \quad (2.41)$$

Writing ω_j in terms of its real and imaginary parts in the last term of equation 2.40, we find that since the real part has odd parity in $\mathbf{k}$, it vanishes upon integrating. So, only the imaginary part of ω_j' survives in the second term of equation 2.40, and this term can be written

$$\sum_j \frac{e^{2\gamma_j t}}{\pi\,\gamma_j\left|\frac{\partial\hat{\varepsilon}(\mathbf{k},\omega)}{\partial\omega}\right|^2_{\omega_j}}\left[\frac{\gamma_j}{(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v})^2+\gamma_j^2}\right]\left[\frac{\gamma_j}{(\omega_{R,j}-\mathbf{k}\cdot\mathbf{v}')^2+\gamma_j^2}\right]. \quad (2.42)$$

Thus, the collisional kernel can be written as the sum of two terms: $\mathcal{Q} = \mathcal{Q}_{\rm LB} + \mathcal{Q}_{\rm IE}$. The first is the Lenard-Balescu term

$$\boxed{\mathcal{Q}_{\rm LB} = \frac{2q_s^2q_{s'}^2}{m_s}\int d^3k\frac{\mathbf{kk}}{k^4}\frac{\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]}{\left|\hat{\varepsilon}(\mathbf{k},\mathbf{k}\cdot\mathbf{v})\right|^2}} \quad (2.43)$$

that describes the conventional Coulomb scattering of individual particles that are Debye shielded due to the plasma polarization. The second is the instability-enhanced term

$$\boxed{\mathcal{Q}_{\mathrm{IE}} = \frac{2q_s^2 q_{s'}^2}{\pi m_s} \int d^3k \frac{\mathbf{kk}}{k^4} \sum_j \frac{\gamma_j \, \exp\left(2\gamma_j t\right)}{[(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v})^2 + \gamma_j^2][(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v}') + \gamma_j^2] \left|\partial\hat{\varepsilon}(\mathbf{k},\omega)/\partial\omega\right|^2_{\omega_j}}} \tag{2.44}$$

that describes the scattering of particles by collective fluctuations that arise due to discrete particle motion, and become amplified due to the dielectric nature of the plasma. In the stable plasma limit, $\gamma_j < 0$, the instability-enhanced interaction term rapidly decays and is entirely negligible, thus returning the Lenard-Balescu equation. The plasma kinetic equation is then $df_s/dt = \sum_{s'} C(f_s, f_{s'})$ where

$$C(f_s, f_{s'}) = -\frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v' \; \mathcal{Q} \cdot \left(\frac{1}{m_{s'}} \frac{\partial}{\partial \mathbf{v}'} - \frac{1}{m_s} \frac{\partial}{\partial \mathbf{v}} \right) f_s(\mathbf{v}) f_{s'}(\mathbf{v}'), \tag{2.45}$$

with $\mathcal{Q}$ given by equations 2.43 and 2.44. We will discuss further how the $\exp(2\gamma_j t)$ term needs to be evaluated in the reference frame of the unstable waves (for convective instabilities) in section 2.4.

The small $\mathbf{k}$ integration limit (corresponding to large b) is resolved in the Lenard-Balescu equation 2.43 because it accounts for plasma polarization, i.e., Debye shielding. This is a main result of the generalization that the Lenard-Balescu equation provides over the Landau and Rosenbluth equations from section 1.1. However, the integral logarithmically diverges in the large $\mathbf{k}$ limit because we have not properly accounted for large-angle scattering when two point particles are in very close proximity to one another. This was accounted for in the Boltzmann approach, from section 1.13, which showed that the appropriate cutoff is $1/b_{\min}$ where $b_{\min}$ is the minimum impact parameter of equation 1.41. If necessary, the same cutoff would also be appropriate for the instability-enhanced term of equation 2.44 because in either case it describes the interaction between individual particles which is limited in closeness by $b_{\min}$. However, equation 2.44 typically does not diverge in either the large or small $\mathbf{k}$ limit because waves are stabilized in these limits; so no cutoff is required. Wave damping mechanisms typically exist for large $\mathbf{k}$ that effectively truncate the upper limit of integration at a value smaller than $1/b_{\min}$.

A simplification of equation 2.44 can also be formed for the very common case of weakly growing instabilities, which satisfy $\gamma_j \ll |\omega_{R,j} - \mathbf{k}\cdot\mathbf{v}|$. In this case, the $\mathbf{v}$ and $\mathbf{v}'$ terms of equation 2.44 can be approximated using the Lorentzian representation for a Dirac delta function

$$\frac{\Delta}{x^2 + \Delta^2} \approx \pi\delta(x) \;\; \text{if} \;\; \frac{\Delta}{x} \ll 1. \tag{2.46}$$

Applying this approximation to equation 2.44 gives the expression

$$\mathcal{Q}_{\mathrm{IE}} \approx \sum_j \frac{2q_s^2 q_{s'}^2}{m_s} \int d^3k \frac{\mathbf{kk}}{k^4} \frac{\pi\, \delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]\, \delta(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v}) \exp(2\gamma_j t)}{\gamma_j \left|\partial\hat{\varepsilon}(\mathbf{k},\omega)/\partial\omega\right|^2_{\omega_j}}. \tag{2.47}$$

We will see in section 3.4.7 that equation 2.47 can be very useful for determining the equilibrium state of a weakly unstable plasma.

An alternative, but equivalent, form for the kernel $\mathcal{Q} = \mathcal{Q}_{\mathrm{LB}} + \mathcal{Q}_{\mathrm{IE}}$ (from equations 2.43 and 2.47) is

$$\mathcal{Q} \;= \frac{q_s^2}{m_s} \int \frac{d^3k}{(2\pi)^3}\, \delta_i\tilde{\mathbf{E}}(\mathbf{k},t)\, \delta_i\tilde{\mathbf{E}}(\mathbf{k},t)\, \delta\big[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')\big] \tag{2.48}$$

where $\delta_i\tilde{\mathbf{E}}$ is the inverse Laplace transform of equation 2.23. The equivalence of equations 2.48 and $\mathcal{Q}$ from equations 2.43 and 2.44 can be checked by an analysis similar to what is provided above, including the neglect of rapidly oscillating "cross" terms in $\mathbf{k}$ space, but without the ensemble average. This alternative form for $\mathcal{Q}$ shows explicitly that it is the "discrete particle" electric fields around individual particles that causes scattering. When instabilities are not present, these fields are the usual Coulomb fields of the charged particles Debye shielded due to plasma polarization. In this case, scattering is effectively limited to particles within a Debye sphere of each other. The presence of instabilities, however, gives rise to a longer range interaction between particles mediated by waves excited through the plasma dielectric. In this manner, scattering between two particles can reach well beyond a Debye sphere.

2.2 BBGKY Hierarchy Approach

In section 2.1 we used the Klimontovich equation which accounted for each particle individually in a six-dimensional position-velocity phase space. Using the test particle approach, which followed the trajectory of each particle and appropriately averaged the exact distribution, a plasma kinetic equation was derived. In this section, we use an approach based on the Liouville equation. Instead of considering each particle individually, the Liouville equation considers all N particles in the plasma to be represented by a single point in a $6N$-dimensional phase space. The phase space consists of the position and velocity of each particle. In analogy to the density of particles, F, in the 6-dimensional phase space from section 2.1, we will now be concerned with the density of systems, D_N, in a 6N-dimensional phase space.

Reduced distribution functions f_i will be defined as integrals of D_N over $6(N-i)$ dimensions of the phase space. The evolution equations of these reduced distribution functions constitutes the BBGKY hierarchy, which is named after Bogoliubov [59], Born [60], Green [60], Kirkwood [61, 62], and Yvon [63]. Ultimately, the evolution equation of f_1 is the lowest-order plasma kinetic equation that we are interested in. However, an exact solution of the f_1 equation of the BBGKY hierarchy requires the solution of the f_2 equation which, in turn, requires the f_3 solution and so on for all of the f_i equations in the hierarchy; it is not a closed set of equations. The Mayer cluster expansion [64] provides a method for relating the f_i and leads to the formulation for a truncation scheme.

After applying the truncation suggested by the Mayer cluster expansion, the BBGKY hierarchy reduces to two equations; one for f_1, which is analogous to f from section 2.1, and one for P which is the pair correlation and is analogous to δf from section 2.1. A solution to the P evolution equation is obtained and leads to a collision operator for the f_1 equation that is equivalent to the one derived in section 2.1. Like section 2.1, the new part of this derivation is to allow for instabilities; the Lenard-Balescu term has been derived from the BBGKY hierarchy before [22, 33, 57].

2.2.1 The Liouville Equation and BBGKY Hierarchy

Consider a plasma with N particles. The present dynamical state of the plasma, referred to here as a system, is the point $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N; \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N)$ which we will denote as $(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_N)$ where $\mathbf{X}_i = (\mathbf{x}_i, \mathbf{v}_i)$ is a six-dimensional phase-space vector. The location of this point is, of course, dependent on time as the individual particles move around. The individual particle trajectories are

$$\frac{d\mathbf{x}_i(t)}{dt} = \mathbf{v}_i(t) \quad \text{and} \quad \frac{d\mathbf{v}_i(t)}{dt} = \mathbf{a}_i. \tag{2.49}$$

Let $D_N(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_N) \geq 0$ denote the probability distribution function of the system in the 6N-dimensional phase space. We assume that no states are created or destroyed, so the system evolves from one phase space position $[\mathbf{X}_1(t_1), \mathbf{X}_2(t_1), \ldots, \mathbf{X}_N(t_1)]$ to another $[\mathbf{X}_1(t_2), \mathbf{X}_2(t_2), \ldots, \mathbf{X}_N(t_2)]$ in time. Thus, the probability of finding the system in some given state must be conserved

$$D_N[\mathbf{X}_1(0), \ldots, \mathbf{X}_N(0)] d^6X_1(0) \ldots d^6X_N(0) = D_N[\mathbf{X}_1(t), \ldots, \mathbf{X}_N(t)] d^6X_1(t) \ldots d^6X_N(t). \tag{2.50}$$

Since the phase space coordinates themselves do not depend on time, $d^6X_i(t) = d^6X_i(0)$, the probability

distribution satisfies

$$D_N[\mathbf{X}_1(0), \mathbf{X}_2(0) \ldots, \mathbf{X}_N(0)] = D_N[\mathbf{X}_1(t), \mathbf{X}_2(t) \ldots, \mathbf{X}_N(t)]. \tag{2.51}$$

This is called *Liouville's theorem*, see e.g. [33]; it states that the probability distribution function for the system, D_N, is constant along the path that the system follows in phase space. Taking the total time derivative of equation 2.51 gives

$$\underbrace{\frac{d}{dt} D_N[\mathbf{X}_1(0), \ldots, \mathbf{X}_N(0)]}_{=0} = \frac{d}{dt} D_N[\mathbf{X}_1(t), \ldots, \mathbf{X}_N(t)]. \tag{2.52}$$

Using the chain rule, the total time derivative can be written

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial}{\partial t} + \frac{\partial \mathbf{X}_1}{\partial t} \cdot \frac{\partial}{\partial \mathbf{X}_1} + \cdots + \frac{\partial \mathbf{X}_N}{\partial t} \cdot \frac{\partial}{\partial \mathbf{X}_N} \\ &= \frac{\partial}{\partial t} + \sum_{i=1}^{N} \left[\mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} + \mathbf{a}_i \cdot \frac{\partial}{\partial \mathbf{v}_i} \right]. \end{aligned} \tag{2.53}$$

Putting equation 2.53 into equation 2.52 gives the evolution equation for the probability density of the system, called the *Liouville equation* [1] [33],

$$\frac{\partial D_N}{\partial t} + \sum_{i=1}^{N} \left[\mathbf{v}_i \cdot \frac{\partial D_N}{\partial \mathbf{x}_i} + \mathbf{a}_i \cdot \frac{\partial D_N}{\partial \mathbf{v}_i} \right] = \frac{dD_N}{dt} = 0. \tag{2.54}$$

Applying the Coulomb approximation, we assume no applied electric or magnetic fields, and neglect the magnetic fields produced by charged particle motion. For more on the effects of equilibrium fields, see appendix B. Since we only consider forces due to the electrostatic interaction between particles, the acceleration vector can be identified as

$$\mathbf{a}_i = \sum_{j, j \neq i} \mathbf{a}_{ij}(\mathbf{x}_i - \mathbf{x}_j) = \sum_{j, j \neq i} \frac{q_i \, q_j}{m_i} \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}. \tag{2.55}$$

Next, to form the BBGKY hierarchy, we define the following *reduced probability distributions* [33]

$$f_\alpha(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_\alpha, t) \equiv N^\alpha \int d^6 X_{\alpha+1} \ldots d^6 X_N \, D_N(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_N). \tag{2.56}$$

[1] This equation has been attributed to Liouville in essentially every statistical mechanics textbook (and article) to this day. However, Nolte has recently made a strong case that this is a missattribution [65]. Liouville's contribution [66] was purely mathematical and was related to taking derivatives of the general form of equation 2.53. He did not apply the rule to a physical system. Nolte [65] makes the point that physical applications were introduced later by such luminaries as Fermat, Boltzmann and Poincaré, and that Boltzmann deserves primary credit for it's application in statistical mechanics.

Equation α of the BBGKY hierarchy of equations is formed by integrating equation 2.54 over the $6(N-\alpha)$ phase-space coordinates, $\int d^6X_{\alpha+1}\ldots d^6X_N$:

$$\int d^6X_{\alpha+1}\ldots d^6X_N\left[\frac{\partial D_N}{\partial t}+\sum_{i=1}^{N}\left(\mathbf{v}_i\cdot\frac{\partial D_N}{\partial \mathbf{x}_i}\mathbf{a}_i\cdot\frac{\partial D_N}{\partial \mathbf{v}_i}\right)\right]=0. \tag{2.57}$$

Consider each of the three terms individually. The first is simply

$$\int d^6X_{\alpha+1}\ldots d^6X_N\,\frac{\partial D_N}{\partial t}=\frac{1}{N^\alpha}\frac{\partial f_\alpha}{\partial t}. \tag{2.58}$$

The second term is

$$\begin{aligned}
&\int d^6X_{\alpha+1}\ldots d^6X_N\sum_{i=1}^{N}\mathbf{v}_i\cdot\frac{\partial D_N}{\partial \mathbf{x}_i} \qquad (2.59)\\
&\quad=\int d^6X_{\alpha+1}\ldots d^6X_N\sum_{i=1}^{\alpha}\mathbf{v}_i\cdot\frac{\partial D_N}{\partial \mathbf{x}_i}+\int d^6X_{\alpha+1}\ldots d^6X_N\sum_{i=\alpha+1}^{N}\mathbf{v}_i\cdot\frac{\partial D_N}{\partial \mathbf{x}_i}\\
&\quad=\sum_{i=1}^{\alpha}\mathbf{v}_i\cdot\frac{\partial}{\partial \mathbf{x}_i}\underbrace{\int d^6X_{\alpha+1}\ldots d^6X_N D_N}_{=f_\alpha/N^\alpha}+\sum_{i=\alpha+1}^{N}\underbrace{\int d^6X_{\alpha+1}\ldots d^6X_N\frac{\partial}{\partial \mathbf{x}_i}\cdot\left(\mathbf{v}_i D_N\right)}_{=0}\\
&\quad=\frac{1}{N^\alpha}\sum_{i=1}^{\alpha}\mathbf{v}_i\cdot\frac{\partial f_\alpha}{\partial \mathbf{x}_i},
\end{aligned}$$

in which the surface integral term vanishes by the assumption that there are no particles at the infinitely distant boundaries. For the third term, we first note that $\mathbf{a}_i$ does not depend on $\mathbf{v}_i$, then we find

$$\begin{aligned}
&\int d^6X_{\alpha+1}\ldots d^6X_N\sum_{i=1}^{N}\mathbf{a}_i\cdot\frac{\partial D_N}{\partial \mathbf{v}_i} \qquad (2.60)\\
&\quad=\int d^6X_{\alpha+1}\ldots d^6X_N\sum_{i=1}^{\alpha}\mathbf{a}_i\cdot\frac{\partial D_N}{\partial \mathbf{v}_i}+\underbrace{\int d^6X_{\alpha+1}\ldots d^6X_N\sum_{i=\alpha+1}^{N}\frac{\partial}{\partial \mathbf{v}_i}\cdot\left(\mathbf{a}_i D_N\right)}_{=0}\\
&\quad=\sum_{i=1}^{\alpha}\frac{\partial}{\partial \mathbf{v}_i}\cdot\int d^6X_{\alpha+1}\ldots d^6X_N\left(\sum_{j=1}^{\alpha}\mathbf{a}_{ij}+\sum_{i=1+\alpha}^{N}\mathbf{a}_{ij}\right)D_N\\
&\quad=\frac{1}{N^\alpha}\sum_{i=1}^{\alpha}\sum_{j=1}^{\alpha}\mathbf{a}_{ij}\cdot\frac{\partial f_\alpha}{\partial \mathbf{v}_i}+\sum_{i=1}^{\alpha}\frac{\partial}{\partial \mathbf{v}_i}\cdot(N_\alpha)\int d^6X_{\alpha+1}\mathbf{a}_{i,\alpha+1}\int d^6X_{\alpha+2}\ldots d^6X_N\,D_N\\
&\quad=\frac{1}{N^\alpha}\sum_{i=1}^{\alpha}\sum_{j=1}^{\alpha}\mathbf{a}_{ij}\cdot\frac{\partial f_\alpha}{\partial \mathbf{v}_i}+\frac{(N-\alpha)}{N^{\alpha+1}}\sum_{i=1}^{\alpha}\int d^6X_{\alpha+1}\mathbf{a}_{i,\alpha+1}\cdot\frac{\partial f_{\alpha+1}}{\partial \mathbf{v}_i}.
\end{aligned}$$

Putting the results of equations 2.58, 2.59 and 2.60, back into equation 2.57, we find that equation α of the BBGKY hierarchy is

$$\boxed{\frac{\partial f_\alpha}{\partial t}+\sum_{i=1}^{\alpha}\mathbf{v}_i\cdot\frac{\partial f_\alpha}{\partial \mathbf{x}_i}+\sum_{i=1}^{\alpha}\sum_{j=1}^{\alpha}\mathbf{a}_{ij}\cdot\frac{\partial f_\alpha}{\partial \mathbf{v}_i}+\frac{N-\alpha}{N}\sum_{i=1}^{\alpha}\int d^6\mathbf{X}_{\alpha+1}\mathbf{a}_{i,\alpha+1}\cdot\frac{\partial f_{\alpha+1}}{\partial \mathbf{v}_i}=0.} \tag{2.61}$$

The evolution equation for each f_α depends on $f_{\alpha+1}$, so we need a truncation scheme in order to solve for any of the reduced probability distributions (we are interested in f_1, which represents the lowest-order smoothed distribution function). Thus, a closure scheme is required to solve equation 2.61.

2.2.2 Plasma Kinetic Equation

In the two-body interaction approximation, the reduced distribution functions $f_1, f_2, \ldots f_3$ would be statistically independent and we could write the two-particle distribution as the product of single-particle distributions $f_2(\mathbf{X}_1, \mathbf{X}_2) = f_1(\mathbf{X}_1)f_1(\mathbf{X}_2)$. Because particles in a plasma do not only interact in a two-body fashion, we want to account for collective effects as well. So, we write the two particle distribution function in terms of the sum of the statistically independent part and a pair correlation. Continuing this process for multiple particle correlations leads to the *Mayer cluster expansion* [64]:

$$\begin{aligned} f_1(\mathbf{X}_1) &= f(\mathbf{X}_1), \\ f_2(\mathbf{X}_1, \mathbf{X}_2) &= f(\mathbf{X}_1)f(\mathbf{X}_2) + P_{1,2}(\mathbf{X}_1, \mathbf{X}_2), \\ f_3(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) &= f(\mathbf{X}_1)f(\mathbf{X}_2)f(\mathbf{X}_3) + f(\mathbf{X}_1)P_{2,3}(\mathbf{X}_2, \mathbf{X}_3) \\ &\quad + f(\mathbf{X}_2)P_{1,3}(\mathbf{X}_1, \mathbf{X}_3) + f(\mathbf{X}_3)P_{1,2}(\mathbf{X}_1, \mathbf{X}_2) + T(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3), \end{aligned} \tag{2.62}$$

which can continue to f_N. P is called the *pair correlation* and T is the *triplet correlation.*

An essential feature of the cluster expansion is that the higher order correlations have smaller and smaller contribution to the evolution equation of f_1. In fact, it can be shown that in a stable plasma $T/(fP) \sim \mathcal{O}(\Lambda^{-1})$ where $\Lambda \sim n\lambda_D^3$ is the plasma parameter. For a detailed discussion of this, see chapter 8 of reference [57]. The truncation scheme we apply is simply that $T = 0$. In unstable plasmas, such as we consider here, the small parameter characterizing higher-order terms becomes Λ^{-1} times the amplification of collisions due to instabilities. After the instability amplitude becomes too large, this parameter is no longer small and T, as well as higher order terms, must be included. Some nonlinear effects, such as mode coupling, enter the hierarchy at the triplet correlation T level [5].

Recalling from equation 2.55 that $\mathbf{a}_{i,i} = 0$ and putting the Mayer cluster expansion of equation 2.62 into the $\alpha = 1$ equation of the BBGKY hierarchy of equation 2.61 gives

$$\frac{\partial f(\mathbf{X}_1)}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f(\mathbf{X}_1)}{\partial \mathbf{x}_1} + \mathbf{a}_1 \cdot \frac{\partial f(\mathbf{X}_1)}{\partial \mathbf{v}_1} = -\int d^6X_2\, \mathbf{a}_{1,2} \cdot \frac{\partial P_{1,2}}{\partial \mathbf{v}_1}, \tag{2.63}$$

which is the lowest order kinetic equation and the right side is the collision operator that we will solve for using the $\alpha = 2$ equation. In equation 2.63 we have used the notation $P_{i,j} = P(\mathbf{X}_i, \mathbf{X}_j)$ and have identified

$$\mathbf{a}_i(\mathbf{x}_i, t) \equiv \int d^6X_j \, \mathbf{a}_{i,j} \, f(\mathbf{X}_j, t), \tag{2.64}$$

which is an average of the electrostatic fields surrounding individual particles (equation 2.55 for a continuous charge distribution). Also, since $N \gg \alpha$ we have assumed that $(N - \alpha)/N \approx 1$. We will also use this for the $\alpha = 2$ equation.

Solving the $\alpha = 2$ equation

$$\underbrace{\frac{\partial f_2}{\partial t}}_{(1)} + \underbrace{\mathbf{v}_1 \cdot \frac{\partial f_2}{\partial \mathbf{x}_1} + \mathbf{v}_2 \cdot \frac{\partial f_2}{\partial \mathbf{x}_2}}_{(2)} + \underbrace{\mathbf{a}_{1,2} \cdot \frac{\partial f_2}{\partial \mathbf{v}_1} + \mathbf{a}_{2,1} \cdot \frac{\partial f_2}{\partial \mathbf{v}_2}}_{(3)} + \underbrace{\int d^6X_3 \, \mathbf{a}_{1,3} \cdot \frac{\partial f_3}{\partial \mathbf{v}_1} + \int d^6X_3 \, \mathbf{a}_{2,3} \cdot \frac{\partial f_3}{\partial \mathbf{v}_2}}_{(4)} = 0 \tag{2.65}$$

is a bit more involved. Putting in the cluster expansion with $T = 0$ into each of these terms gives

$$(1) = \underbrace{f(\mathbf{X}_1)\frac{\partial f(\mathbf{X}_2)}{\partial t}}_{(e)} + \underbrace{f(\mathbf{X}_2)\frac{\partial f(\mathbf{X}_1)}{\partial t}}_{(a)} + \frac{\partial P_{1,2}}{\partial t}, \tag{2.66}$$

$$(2) = \underbrace{f(\mathbf{X}_2)\, \mathbf{v}_1 \cdot \frac{\partial f(\mathbf{X}_1)}{\partial \mathbf{x}_1}}_{(b)} + \mathbf{v}_1 \cdot \frac{\partial P_{1,2}}{\partial \mathbf{x}_1} + \underbrace{f(\mathbf{X}_1)\, \mathbf{v}_2 \cdot \frac{\partial f(\mathbf{X}_2)}{\partial \mathbf{x}_2}}_{(f)} + \mathbf{v}_2 \cdot \frac{\partial P_{1,2}}{\partial \mathbf{x}_2}, \tag{2.67}$$

$$(3) = f(\mathbf{X}_2)\, \mathbf{a}_{1,2} \cdot \frac{\partial f(\mathbf{X}_1)}{\partial \mathbf{v}_1} + \mathbf{a}_{1,2} \cdot \frac{\partial P_{1,2}}{\partial \mathbf{v}_1} + f(\mathbf{X}_1)\, \mathbf{a}_{2,1} \cdot \frac{\partial f(\mathbf{X}_2)}{\partial \mathbf{v}_2} + \mathbf{a}_{2,1} \cdot \frac{\partial P_{1,2}}{\partial \mathbf{v}_2}, \tag{2.68}$$

$$\begin{aligned}(4) = \int d^6X_3 \Bigg[& \underbrace{f(\mathbf{X}_2) f(\mathbf{X}_3) \mathbf{a}_{1,3} \cdot \frac{\partial f(\mathbf{X}_1)}{\partial \mathbf{v}_1}}_{(c)} + P_{2,3} \mathbf{a}_{1,3} \cdot \frac{\partial f(\mathbf{X}_1)}{\partial \mathbf{v}_1} + \underbrace{f(\mathbf{X}_2)\, \mathbf{a}_{1,3} \cdot \frac{\partial P_{1,3}}{\partial \mathbf{v}_1}}_{(d)} + f(\mathbf{X}_3)\, \mathbf{a}_{1,3} \cdot \frac{\partial P_{1,2}}{\partial \mathbf{v}_1} \\ & + \underbrace{f(\mathbf{X}_1) f(\mathbf{X}_3)\, \mathbf{a}_{2,3} \cdot \frac{\partial f(\mathbf{X}_2)}{\partial \mathbf{v}_2}}_{(g)} + \underbrace{f(\mathbf{X}_1) \mathbf{a}_{2,3} \cdot \frac{\partial P_{2,3}}{\partial \mathbf{v}_2}}_{(h)} + P_{1,3}\, \mathbf{a}_{2,3} \cdot \frac{\partial f(\mathbf{X}_2)}{\partial \mathbf{v}_2} + f(\mathbf{X}_3)\, \mathbf{a}_{2,3} \cdot \frac{\partial P_{1,2}}{\partial \mathbf{v}_2} \Bigg].\end{aligned} \tag{2.69}$$

But, from equation 2.63, we find that $(a) + (b) + (c) + (d) = 0$ and $(e) + (f) + (g) + (h) = 0$. If we also apply equation 2.64 in term (4) where

$$\int d^6X_3 \, \mathbf{a}_{1,3} f(\mathbf{X}_3) = \mathbf{a}_1 \quad \text{and} \quad \int d^6X_3 \, \mathbf{a}_{2,3} f(\mathbf{X}_3) = \mathbf{a}_2, \tag{2.70}$$

equation 2.65 reduces to

$$\left[\frac{\partial}{\partial t}+\sum_{i=1}^{2}\left(\mathbf{v}_i\cdot\frac{\partial}{\partial \mathbf{x}_i}+\mathbf{a}_i\cdot\frac{\partial}{\partial \mathbf{v}_i}+\sum_{\substack{j=1\\ j\neq i}}^{2}\mathbf{a}_{i,j}\cdot\frac{\partial}{\partial \mathbf{v}_i}\right)\right]P_{1,2}+\sum_{i=1}^{2}\sum_{\substack{j=1\\ j\neq i}}^{2}\frac{\partial f(\mathbf{X}_i)}{\partial \mathbf{v}_i}\cdot\int d^6X_3\,\mathbf{a}_{i,3}P_{j,3} \qquad (2.71)$$
$$=-\sum_{i=1}^{2}\sum_{\substack{j=1\\ j\neq i}}^{2}\mathbf{a}_{i,j}\cdot\frac{\partial}{\partial \mathbf{v}_i}f(\mathbf{X}_i)f(\mathbf{X}_j).$$

We assume that acceleration due to ensemble averaged forces (i.e., equation 2.64), which are from potential variations over macroscopic spatial scales, are small. Thus the $\mathbf{a}_i\cdot\partial/\partial\mathbf{v}_i$ terms in equations 2.63 and 2.71 can be neglected. Also, the $\mathbf{a}_{i,j}\cdot\partial/\partial\mathbf{v}_i$ terms on the left side of equation 2.71 can be neglected because they are Λ^{-1} smaller than the $\partial/\partial t+\mathbf{v}_i\cdot\partial/\partial\mathbf{x}_i$ terms. This scaling can be obtained by putting $\Delta x\sim\lambda_D$ into equation 2.55, which gives

$$a_{i,j}\frac{\partial/\partial\mathbf{v}_i}{\partial/\partial t}\sim\frac{e^2}{m\lambda_D^2}\frac{1/v_T}{\omega_p}\sim\Lambda^{-1}. \qquad (2.72)$$

Also, since we only consider electrostatic interactions between particles, $P_{i,j}(\mathbf{x}_i,\mathbf{x}_j)=P_{i,j}(\mathbf{x}_i-\mathbf{x}_j)$.

With these approximations the plasma kinetic equation becomes

$$\frac{\partial f(\mathbf{X}_1)}{\partial t}+\mathbf{v}_1\cdot\frac{\partial f(\mathbf{X}_1)}{\partial \mathbf{x}_1}=-\int d^6X_2\,\mathbf{a}_{1,2}\cdot\frac{\partial P_{1,2}}{\partial \mathbf{v}_1} \qquad (2.73)$$

and the pair correlation equation is

$$\frac{\partial P_{1,2}}{\partial t}+\sum_{i=1}^{2}\mathbf{v}_i\cdot\frac{\partial P_{1,2}}{\partial \mathbf{x}_i}+\sum_{i=1}^{2}\sum_{\substack{j=1\\ j\neq i}}^{2}\frac{\partial f(\mathbf{X}_i)}{\partial \mathbf{v}_i}\cdot\int d^6X_3\,\mathbf{a}_{i,3}P_{j,3}=-\sum_{i=1}^{2}\sum_{\substack{j=1\\ j\neq i}}^{2}\mathbf{a}_{i,j}\cdot\frac{\partial}{\partial \mathbf{v}_i}f(\mathbf{X}_i)f(\mathbf{X}_j). \qquad (2.74)$$

Next, we apply the Bogoliubov hypothesis: the characteristic time and spatial scales for relaxation of the pair correlation P are much shorter than that for f [59]. We denote the longer time and spatial scales $(\bar{\mathbf{x}},\bar{t})$ and Fourier transform $(\mathcal{F})$ with respect to the shorter spatial scales on which f is approximately constant. We use the same Fourier transform definition as equation 2.14: $\mathcal{F}\{g(\mathbf{x})\}=\hat{g}(\mathbf{k})=\int d^3x\exp(-i\mathbf{k}\cdot\mathbf{x})g(\mathbf{x})$ with inverse $g(\mathbf{x})=(2\pi)^{-3}\int d^3k\exp(i\mathbf{k}\cdot\mathbf{x})\hat{g}(\mathbf{k})$. The double Fourier transform is then

$$\mathcal{F}_{1,2}\{h(\mathbf{x}_1,\mathbf{x}_2)\}=\hat{h}(\mathbf{k}_1,\mathbf{k}_2)=\int d^3x_1d^3x_2\,e^{-i(\mathbf{k}_1\cdot\mathbf{x}_1+\mathbf{k}_2\cdot\mathbf{x}_2)}h(\mathbf{x}_1,\mathbf{x}_2). \qquad (2.75)$$

Before applying the Fourier transform to equations 2.73 and 2.74, it is useful to note the following three properties for the transform of arbitrary functions h_1 and h_2:

(P1): $\mathcal{F}_{1,2}\{h(\mathbf{x}_1-\mathbf{x}_2)\}=(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2)\hat{h}(\mathbf{k}_1)$.

Proof:

$$\mathcal{F}_{1,2}\big\{h(\mathbf{x}_1-\mathbf{x}_2)\big\}=\int d^3x_1\,d^3x_2\,e^{-i(\mathbf{k}_1\cdot\mathbf{x}_1+\mathbf{k}_2\cdot\mathbf{x}_2)}h(\mathbf{x}_1-\mathbf{x}_2). \tag{2.76}$$

Setting $\mathbf{x}\equiv\mathbf{x}_1-\mathbf{x}_2$, so $d^3x=d^3x_1$ yields

$$\mathcal{F}_{1,2}\big\{h(\mathbf{x}_1-\mathbf{x}_2)\big\}=\underbrace{\int d^3x\,e^{-i\mathbf{k}_1\cdot\mathbf{x}}h(\mathbf{x})}_{\hat{h}(\mathbf{k}_1)}\underbrace{\int d^3x_2\,e^{-i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}_2}}_{(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2)}=(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2)\hat{h}(\mathbf{k}_1). \tag{2.77}$$

(P2): $\int d^3x\ h_1(\mathbf{x})h_2(\mathbf{x})=(2\pi)^{-3}\int d^3k_1\ \hat{h}_1(\mathbf{k}_1)\,\hat{h}_2(-\mathbf{k}_1)$.

Proof:

$$\begin{aligned}\int d^3x\,h_1(\mathbf{x})h_2(\mathbf{x})&=\int d^3x\left[\int\frac{d^3k_1}{(2\pi)^3}\,e^{i\mathbf{k}_1\cdot\mathbf{x}}\hat{h}_1(\mathbf{k}_1)\right]\left[\int\frac{d^3k_2}{(2\pi)^3}\,e^{i\mathbf{k}_2\cdot\mathbf{x}}\hat{h}_2(\mathbf{k}_2)\right]\\&=\frac{1}{(2\pi)^6}\int d^3k_1\int d^3k_2\,\hat{h}_1(\mathbf{k}_1)\hat{h}_2(\mathbf{k}_2)\int d^3x\,e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}}\\&=\frac{1}{(2\pi)^3}\int d^3k_1\,\hat{h}_1(\mathbf{k}_1)\int d^3k_2\,\hat{h}_2(\mathbf{k}_2)\,\delta(\mathbf{k}_1+\mathbf{k}_2)\\&=\frac{1}{(2\pi)^3}\int d^3k_1\,\hat{h}_1(\mathbf{k}_1)\,\hat{h}_2(-\mathbf{k}_1).\end{aligned} \tag{2.78}$$

(P3): $\mathcal{F}_{1,2}\{\int d^3x_3\ h_1(\mathbf{x}_1-\mathbf{x}_3)h_2(\mathbf{x}_2-\mathbf{x}_3)\}=(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2)\hat{h}_1(\mathbf{k}_1)\hat{h}_2(\mathbf{k}_2)$.

Proof:

$$\mathcal{F}_{1,2}\bigg\{\int d^3x_3h_1(\mathbf{x}_1-\mathbf{x}_3)h_2(\mathbf{x}_2-\mathbf{x}_3)\bigg\}=\int d^3x_3d^3x_1d^3x_2e^{-i(\mathbf{k}_1\cdot\mathbf{x}_1+\mathbf{k}_2\cdot\mathbf{x}_2)}h_1(\mathbf{x}_1-\mathbf{x}_3)h_2(\mathbf{x}_2-\mathbf{x}_3), \tag{2.79}$$

let $\mathbf{u}\equiv\mathbf{x}_1-\mathbf{x}_3$ and $\mathbf{w}\equiv\mathbf{x}_2-\mathbf{x}_3$, then

$$\begin{aligned}\mathcal{F}_{1,2}\{\ldots\}&=\int d^3x_3\,e^{-i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}_3}\int d^3u\,e^{-i\mathbf{k}_1\cdot\mathbf{u}}h_1(\mathbf{u})\int d^3w\,e^{-i\mathbf{k}_2\cdot\mathbf{w}}h_2(\mathbf{w})\\&=(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2)\,\hat{h}_1(\mathbf{k}_1)\,\hat{h}_2(\mathbf{k}_2).\end{aligned} \tag{2.80}$$

With Bogoliubov's hypothesis, equation 2.73 is

$$\left(\frac{\partial}{\partial\bar{t}}+\mathbf{v}_1\cdot\frac{\partial}{\partial\bar{\mathbf{x}}_1}\right)f(\bar{\mathbf{x}}_1,\mathbf{v}_1,\bar{t})=-\frac{\partial}{\partial\mathbf{v}_1}\cdot\int d^3v_2\int d^3x_2\,\mathbf{a}_{1,2}P_{1,2}. \tag{2.81}$$

Recalling from (P2) that

$$\int d^3x_2\,\mathbf{a}_{1,2}(\mathbf{x}_1-\mathbf{x}_2)\,P_{1,2}(\mathbf{x}_1-\mathbf{x}_2)=\frac{1}{(2\pi)^3}\int d^3k_1\,\hat{\mathbf{a}}_{1,2}(-\mathbf{k}_1)\,\hat{P}_{1,2}(\mathbf{k}_1), \tag{2.82}$$

and noting from equation 2.55 that

$$\hat{\mathbf{a}}_{1,2}(\mathbf{k}_1) = \frac{q_1 q_2}{m_1} \frac{4\pi i \mathbf{k}_1}{k_1^2}, \tag{2.83}$$

the plasma kinetic equation is

$$\left(\frac{\partial}{\partial \bar{t}} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \bar{\mathbf{x}}_1} \right) f(\bar{\mathbf{x}}_1, \mathbf{v}_1, \bar{t}) = -\frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{4\pi q_1 q_2}{m_1} \int \frac{d^3 k_1}{(2\pi)^3} \frac{-i\mathbf{k}_1}{k_1^2} \int d^3 v_2 \, \hat{P}_{1,2}(\mathbf{k}_1) \right]. \tag{2.84}$$

Thus we require $\hat{P}_{1,2}$ in order to determine the collision operator. If we apply properties $(P1) - (P3)$ and equation 2.83, we find that equations 2.73 and 2.74 can be written

$$\left(\frac{\partial}{\partial \bar{t}} + \mathbf{v}_1 \cdot \frac{\partial}{\partial \bar{\mathbf{x}}_1} \right) f(\bar{\mathbf{x}}_1, \mathbf{v}_1, \bar{t}) = C(f_1) = -\frac{\partial}{\partial \mathbf{v}_1} \cdot \mathbf{J}_v \tag{2.85}$$

and

$$\left[\frac{\partial}{\partial t} + L_1(\mathbf{k}_1) + L_2(-\mathbf{k}_1) \right] \hat{P}_{1,2}(\mathbf{k}_1, \mathbf{v}_1, \mathbf{v}_2, t) = \hat{S}(\mathbf{k}_1, \mathbf{v}_1, \mathbf{v}_2, \bar{t}). \tag{2.86}$$

Here $\mathbf{J}_v$ is the collisional current

$$\mathbf{J}_v \equiv \frac{4\pi q_1 q_2}{m_1} \int \frac{d^3 k_1}{(2\pi)^3} \frac{-i\mathbf{k}_1}{k_1^2} \int d^3 v_2 \hat{P}_{1,2}(\mathbf{k}_1, \mathbf{v}_1, \mathbf{v}_2, t), \tag{2.87}$$

L_j is the integral operator

$$L_j(\mathbf{k}_1) \equiv i\mathbf{k}_1 \cdot \mathbf{v}_j - i \frac{4\pi q_1 q_2}{m_j} \frac{\mathbf{k}_1}{k_1^2} \cdot \frac{\partial f(\mathbf{v}_j)}{\partial \mathbf{v}_j} \int d^3 v_j \tag{2.88}$$

and $\hat{S}$ is the source term for the pair correlation function equation

$$\hat{S}(\mathbf{k}_1, \mathbf{v}_1, \mathbf{v}_2) = 4\pi i q_1 q_2 \frac{\mathbf{k}_1}{k_1^2} \cdot \left(\frac{1}{m_1} \frac{\partial}{\partial \mathbf{v}_1} - \frac{1}{m_2} \frac{\partial}{\partial \mathbf{v}_2} \right) f(\mathbf{v}_1) f(\mathbf{v}_2). \tag{2.89}$$

We next use equations 2.86 and 2.87 to solve for the collision operator, which is the right side of equation 2.85.

After Laplace transforming with respect to the fast timescale t, equation 2.86 can be written formally as

$$\hat{P}_{1,2}(\mathbf{k}_1, \omega) = \frac{\tilde{P}_{1,2}(\mathbf{k}_1, t=0) - \hat{S}/i\omega}{-i\omega + L_1(\mathbf{k}_1) + L_2(-\mathbf{k}_1)} \tag{2.90}$$

in which the velocity dependence of $\hat{P}_{1,2}$, $\hat{S}$ and L has been suppressed for notational convenience. In the following, we neglect the initial pair correlation term $\tilde{P}_{1,2}(t=0)$ because it is smaller in plasma parameter than the continually evolving collisional source term $\hat{S}$. In Davidson's approach to quasilinear theory, which is a collisionless description of wave-particle interactions, the collisional source term $(\hat{S}/i\omega)$ is neglected [58]. Keeping the initial pair correlation term leads to a diffusion equation [58], which will be discussed in section 3.2. Here we are interested in a collision operator.

2.2.3 A Collision Operator From the Source Term

For the collision operator, we require evaluation of

$$\hat{P}_{1,2}(\mathbf{k}_1,\omega) = -\frac{1}{-i\omega + L_1(\mathbf{k}_1) + L_2(-\mathbf{k}_1)}\frac{\hat{S}(\mathbf{k}_1,\mathbf{v}_1,\mathbf{v}_2)}{i\omega}. \tag{2.91}$$

The $1/[-i\omega + L_1(\mathbf{k}_1) + L_2(-\mathbf{k}_1)]$ part of equation 2.91 is an operator that acts on $\hat{S}/i\omega$. It can be written [5]

$$\begin{aligned}\frac{1}{-i\omega + L_1(\mathbf{k}_1) + L_2(-\mathbf{k}_1)} = \frac{1}{(2\pi)^2}\int_{\mathcal{C}_1}\int_{\mathcal{C}_2}\frac{d\omega_1\ d\omega_2}{-i(\omega-\omega_1-\omega_2)} \\ \cdot\frac{1}{[-i\omega_1 + L_1(\mathbf{k}_1)]}\frac{1}{[-i\omega_2 + L_2(-\mathbf{k}_1)]}\end{aligned} \tag{2.92}$$

in which the contours $\mathcal{C}_1$ and $\mathcal{C}_2$ must be chosen such that $\Im\{\omega\} > \Im\{\omega_1+\omega_2\}$. Frieman and Rutherford [5] showed that

$$\frac{1}{-i\omega_1 + L_1(\mathbf{k}_1)} = \frac{i}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_1}\left\{1 - \frac{4\pi q_1 q_2}{m_1 k_1^2}\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_1)/\partial\mathbf{v}_1}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)}\int\frac{d^3v_1}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_1}\right\} \tag{2.93}$$

in which

$$\hat{\varepsilon}(\mathbf{k}_1,\omega_1) \equiv 1 + \frac{4\pi q_1 q_2}{m_1 k_1^2}\int d^3v_1\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_1)/\partial\mathbf{v}_1}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_1}. \tag{2.94}$$

The equivalent expressions for $1/[-i\omega_2 + L_2(-\mathbf{k}_1)]$ and $\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)$ are obtained by the substitutions $\mathbf{v}_1 \leftrightarrow \mathbf{v}_2$, $\omega_1 \leftrightarrow \omega_2$, $m_1 \leftrightarrow m_2$ and $\mathbf{k}_1 \leftrightarrow -\mathbf{k}_1$.

We can check equation 2.93 by applying the operator $[1 + L_1(\mathbf{k}_1)]$ to it and confirming that the result is unity. Recall from equation 2.88 that

$$-i\omega_1 + L_1(\mathbf{k}_1) = -i\omega + i\mathbf{k}_1\mathbf{v}_1 - i\frac{4\pi q_1 q_2}{m_1}\frac{\mathbf{k}_1}{k_1^2}\cdot\frac{\partial f(\mathbf{v}_1)}{\partial\mathbf{v}_1}\int d^3v_1. \tag{2.95}$$

Thus, we can confirm

$$\begin{aligned}&\big[-i\omega_1 + L_1(\mathbf{k}_1)\big]\left\{\frac{1}{-i\omega_1 + L_1(\mathbf{k}_1)}\right\} \qquad (2.96)\\ &= -\left[i(\omega - \mathbf{k}_1\cdot\mathbf{v}_1) + i\frac{4\pi q_1 q_2}{m_1}\frac{\mathbf{k}_1}{k_1^2}\cdot\frac{\partial f(\mathbf{v}_1)}{\partial\mathbf{v}_1}\int d^3v_1\right]\frac{i}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_1}\left\{1 - \frac{4\pi q_1 q_2}{m_1 k_1^2}\frac{\mathbf{k}_1\partial f/\partial\mathbf{v}_1}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)}\int\frac{d^3v_1}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_1}\right\}\\ &= 1 - \frac{4\pi q_1 q_2}{m_1 k_1^2}\frac{\mathbf{k}_1\cdot\partial f/\partial\mathbf{v}_1}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)}\int\frac{d^3v_1}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_1} + \frac{4\pi q_1 q_2}{m_1 k_1^2}\mathbf{k}_1\cdot\frac{\partial f}{\partial\mathbf{v}_1}\int\frac{d^3v_1}{\omega_1 - \mathbf{k}\cdot\mathbf{v}_1}\\ &- \frac{4\pi q_1 q_2}{m_1 k_1^2}\frac{\mathbf{k}_1}{k_1^2}\cdot\frac{\partial f}{\partial\mathbf{v}_1}\underbrace{\frac{4\pi q_1 q_2}{m_1 k_1^2}\int d^3v_1\frac{\mathbf{k}_1\cdot\partial f/\partial\mathbf{v}_1}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_1}}_{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)-1}\frac{1}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)}\int\frac{d^3v_1}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_1} = 1.\end{aligned}$$

We call $\mathcal{R} \equiv 1/[-i\omega_1 + L_1(\mathbf{k}_1)][-i\omega_2 + L_2(-\mathbf{k}_1)]$ the Frieman-Rutherford operator [5] and require $\mathcal{R}\{\hat{S}\}$. Using equation 2.93, the equivalent form for the $1/[-i\omega_2 + L_2(-\mathbf{k_1})]$ term, and the source term of equation 2.89, produces an expression for $\mathcal{R}\{\hat{S}\}$ of the form

$$\mathcal{R}\{\hat{S}\} = -\frac{[(1)+(2)+(3)+(4)]}{(\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_1)(\omega_2 + \mathbf{k}_1\cdot\mathbf{v}_2)} \tag{2.97}$$

in which each of the numbered pieces consists of two terms. These are

$$(1) = \hat{S} = \underbrace{\frac{i4\pi q_1 q_2}{m_1 k_1^2} f(\mathbf{v}_2)\,\mathbf{k}_1\cdot\frac{\partial f(\mathbf{v}_1)}{\partial \mathbf{v}_1}}_{(a)} + \underbrace{\frac{-i4\pi q_1 q_2}{m_2 k_1^2} f(\mathbf{v}_1)\mathbf{k}_1\cdot\frac{\partial f(\mathbf{v}_2)}{\partial \mathbf{v}_2}}_{(b)}, \tag{2.98}$$

$$\begin{aligned}
(2) &= \frac{4\pi q_1 q_2}{m_2 k_1^2}\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_2)/\partial\mathbf{v}_2}{\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)}\int d^3v_2 \frac{\hat{S}}{\omega_2 + \mathbf{k}_1\cdot\mathbf{v}_2} \\
&= \frac{i(4\pi)^2 q_1^2 q_2^2}{m_1 m_2 k_1^4}\,\mathbf{k}_1\cdot\frac{\partial f(\mathbf{v}_2)}{\partial \mathbf{v}_2}\,\mathbf{k}_1\cdot\frac{\partial f(\mathbf{v}_1)}{\partial \mathbf{v}_1}\frac{1}{\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)}\int d^3v_2\frac{f(\mathbf{v}_2)}{\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2} \\
&+ \frac{i4\pi q_1 q_2}{m_2 k_1^2}\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_2)/\partial\mathbf{v}_2}{\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)} f(\mathbf{v}_1)\underbrace{\frac{4\pi q_1 q_2}{m_2 k_1^2}\int d^3v_2\frac{(-\mathbf{k}_1)\cdot\partial f(\mathbf{v}_2)/\partial\mathbf{v}_2}{\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2}}_{\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)-1} \\
&= \underbrace{\frac{i(4\pi)^2 q_1^2 q_2^2}{m_1 m_2 k_1^4}\mathbf{k}_1\cdot\frac{\partial f(\mathbf{v}_2)}{\partial \mathbf{v}_2}\mathbf{k}_1\cdot\frac{\partial f(\mathbf{v}_1)}{\partial \mathbf{v}_1}\frac{1}{\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)}\int d^3v_2\frac{f(\mathbf{v}_2)}{\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2}}_{(c)} \\
&+ \underbrace{\frac{i4\pi q_1 q_2}{m_2 k_1^2} f(\mathbf{v}_1)\mathbf{k}_1\cdot\frac{\partial f(\mathbf{v}_2)}{\partial \mathbf{v}_2}}_{(d)} + \underbrace{\frac{-i4\pi q_1 q_2}{m_2 k_1^2} f(\mathbf{v}_1)\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_2)/\partial\mathbf{v}_2}{\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)}}_{(e)},
\end{aligned} \tag{2.99}$$

and, after identifying $\hat{\varepsilon}(\mathbf{k}_1,\omega_1)$ in an analogous way,

$$\begin{aligned}
(3) &= -\frac{4\pi q_1 q_2}{m_1 k_1^2}\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_1)/\partial\mathbf{v}_1}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)}\int d^3v_1\frac{\hat{S}}{\omega_1-\mathbf{k}_1\cdot\mathbf{v}_1} \\
&= \underbrace{\frac{i(4\pi)^2 q_1^2 q_2^2}{m_1 m_2 k_1^4}\mathbf{k}_1\cdot\frac{\partial f(\mathbf{v}_2)}{\partial \mathbf{v}_2}\mathbf{k}_1\cdot\frac{\partial f(\mathbf{v}_1)}{\partial \mathbf{v}_1}\frac{1}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)}\int d^3v_1\frac{f(\mathbf{v}_1)}{\omega_1-\mathbf{k}_1\cdot\mathbf{v}_1}}_{(f)} \\
&+ \underbrace{\frac{-i4\pi q_1 q_2}{m_1 k_1^2} f(\mathbf{v}_2)\,\mathbf{k}_1\cdot\frac{\partial f(\mathbf{v}_1)}{\partial \mathbf{v}_1}}_{(g)} + \underbrace{\frac{i4\pi q_1 q_2}{m_1 k_1^2} f(\mathbf{v}_2)\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_1)/\partial\mathbf{v}_1}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)}}_{(h)}.
\end{aligned} \tag{2.100}$$

Making the $\hat{\varepsilon}$ identification twice in (4) yields

$$
\begin{aligned}
(4) &= -\frac{(4\pi)^2 q_1^2 q_2^2}{m_1 m_2 k_1^4} \frac{\mathbf{k}_1 \cdot \partial f(\mathbf{v}_1)/\partial \mathbf{v}_1}{\hat{\varepsilon}(\mathbf{k}_1, \omega_1)} \frac{\mathbf{k}_1 \cdot \partial f(\mathbf{v}_2)/\partial \mathbf{v}_2}{\hat{\varepsilon}(-\mathbf{k}_1, \omega_1)} \int \frac{d^3 v_1}{\omega_1 - \mathbf{k}_1 \cdot \mathbf{v}_1} \int \frac{d^3 v_2}{\omega_2 + \mathbf{k}_1 \cdot \mathbf{v}_2} \hat{S} \\
&= \underbrace{\frac{-i(4\pi)^2 q_1^2 q_2^2}{m_1 m_2 k_1^4} \mathbf{k}_1 \cdot \frac{\partial f(\mathbf{v}_1)}{\partial \mathbf{v}_1} \frac{\mathbf{k}_1 \cdot \partial f(\mathbf{v}_2)/\partial \mathbf{v}_2}{\hat{\varepsilon}(-\mathbf{k}_1, \omega_2)} \int d^3 v_2 \frac{f(\mathbf{v}_2)}{\omega_2 + \mathbf{k}_1 \cdot \mathbf{v}_2}}_{(i)} \\
&+ \underbrace{\frac{i(4\pi)^2 q_1^2 q_2^2}{m_1 m_2 k_1^4} \frac{\mathbf{k}_1 \cdot \partial f(\mathbf{v}_1)/\partial \mathbf{v}_1}{\hat{\varepsilon}(\mathbf{k}_1, \omega_1)} \frac{\mathbf{k}_1 \cdot \partial f(\mathbf{v}_2)/\partial \mathbf{v}_2}{\hat{\varepsilon}(-\mathbf{k}_1, \omega_2)} \int d^3 v_2 \frac{f(\mathbf{v}_2)}{\omega_2 + \mathbf{k}_1 \cdot \mathbf{v}_2}}_{(j)} \\
&+ \underbrace{\frac{-i(4\pi)^2 q_1^2 q_2^2}{m_1 m_2 k_1^4} \frac{\mathbf{k}_1 \cdot \partial f(\mathbf{v}_1)/\partial \mathbf{v}_1}{\hat{\varepsilon}(\mathbf{k}_1, \omega_1)} \mathbf{k}_1 \cdot \frac{\partial f(\mathbf{v}_2)}{\partial \mathbf{v}_2} \int d^3 v_1 \frac{f(\mathbf{v}_1)}{\omega_1 - \mathbf{k}_1 \cdot \mathbf{v}_1}}_{(k)} \\
&+ \underbrace{\frac{i(4\pi)^2 q_1^2 q_2^2}{m_1 m_2 k_1^4} \frac{\mathbf{k}_1 \cdot \partial f(\mathbf{v}_1)/\partial \mathbf{v}_1}{\hat{\varepsilon}(\mathbf{k}_1, \omega_1)} \frac{\mathbf{k}_1 \cdot \partial f(\mathbf{v}_2)/\partial \mathbf{v}_2}{\hat{\varepsilon}(-\mathbf{k}_1, \omega_2)} \int d^3 v_1 \frac{f(\mathbf{v}_1)}{\omega_1 - \mathbf{k}_1 \cdot \mathbf{v}_1}}_{(l)}.
\end{aligned}
\tag{2.101}
$$

With the letter identifications, we find that the four numbered terms can be written as the sum of the twelve lettered terms: $(1) + (2) + (3) + (4) = (a) + \ldots + (l)$. However, $(a) = -(g)$, $(b) = -(d)$, $(c) = -(i)$, and $(f) = -(k)$, so these twelve terms reduce to four $(1)+(2)+(3)+(4) = (e)+(h)+(j)+(l)$, which, when put into equation 2.97, yields

$$
\begin{aligned}
\mathcal{R}\{\hat{S}\} = \frac{-4\pi i q_1 q_2/k_1^2}{(\omega_1 - \mathbf{k}_1 \cdot \mathbf{v}_1)(\omega_2 + \mathbf{k}_1 \cdot \mathbf{v}_2)} \Bigg\{ & \frac{f(\mathbf{v}_2)\, \mathbf{k}_1 \cdot \partial f(\mathbf{v}_1)/\partial \mathbf{v}_1}{m_1 \hat{\varepsilon}(\mathbf{k}_1, \omega_1)} - \frac{f(\mathbf{v}_1)\, \mathbf{k}_1 \cdot \partial f(\mathbf{v}_2)/\partial \mathbf{v}_2}{m_2 \hat{\varepsilon}(-\mathbf{k}_1, \omega_2)} \\
&+ \frac{4\pi q_1 q_2}{m_1 m_2 k_1^2} \frac{\mathbf{k}_1 \cdot \partial f(\mathbf{v}_1)/\partial \mathbf{v}_1}{\hat{\varepsilon}(\mathbf{k}_1, \omega_1)} \frac{\mathbf{k}_1 \cdot \partial f(\mathbf{v}_2)/\partial \mathbf{v}_2}{\hat{\varepsilon}(-\mathbf{k}_1, \omega_2)} \left[\int d^3 v_2 \frac{f(\mathbf{v}_2)}{\omega_2 + \mathbf{k}_1 \cdot \mathbf{v}_2} + \int d^3 v_1 \frac{f(\mathbf{v}_1)}{\omega_1 - \mathbf{k}_1 \cdot \mathbf{v}_1} \right] \Bigg\}.
\end{aligned}
\tag{2.102}
$$

Noting that in the last term we can use

$$
\int d^3 v_2 \frac{f(\mathbf{v}_2)}{\omega_2 + \mathbf{k}_1 \cdot \mathbf{v}_2} + \int d^3 v_1 \frac{f(\mathbf{v}_1)}{\omega_1 - \mathbf{k}_1 \cdot \mathbf{v}_1} = (\omega_1 + \omega_2) \int d^3 v_2 \frac{f(\mathbf{v}_2)}{(\omega_2 + \mathbf{k}_1 \cdot \mathbf{v}_2)(\omega_1 - \mathbf{k}_1 \cdot \mathbf{v}_2)},
\tag{2.103}
$$

the Frieman-Rutherford operator acting on the source $\hat{S}$ is

$$
\begin{aligned}
\mathcal{R}\{\hat{S}\} = \frac{-4\pi i q_1 q_2/k_1^2}{(\omega_1 - \mathbf{k}_1 \cdot \mathbf{v}_1)(\omega_2 + \mathbf{k}_1 \cdot \mathbf{v}_2)} \Bigg\{ & \frac{f(\mathbf{v}_2)\, \mathbf{k}_1 \cdot \partial f(\mathbf{v}_1)/\partial \mathbf{v}_1}{m_1 \hat{\varepsilon}(\mathbf{k}_1, \omega_1)} - \frac{f(\mathbf{v}_1)\, \mathbf{k}_1 \cdot \partial f(\mathbf{v}_2)/\partial \mathbf{v}_2}{m_2 \hat{\varepsilon}(-\mathbf{k}_1, \omega_2)} \\
&+ \frac{4\pi q_1 q_2}{m_1 m_2 k_1^2} \frac{\mathbf{k}_1 \cdot \partial f(\mathbf{v}_1)/\partial \mathbf{v}_1}{\hat{\varepsilon}(\mathbf{k}_1, \omega_1)} \frac{\mathbf{k}_1 \cdot \partial f(\mathbf{v}_2)/\partial \mathbf{v}_2}{\hat{\varepsilon}(-\mathbf{k}_1, \omega_2)} \int d^3 v_2 \frac{f(\mathbf{v}_2)(\omega_1 + \omega_2)}{(\omega_2 + \mathbf{k}_1 \cdot \mathbf{v}_2)(\omega_1 - \mathbf{k}_1 \cdot \mathbf{v}_2)} \Bigg\}.
\end{aligned}
\tag{2.104}
$$

For $\mathbf{J}_v$ in equation 2.87, we need $\int d^3 v_2 \hat{P}_{1,2}(\mathbf{k}_1, t)$ which is

$$
\begin{aligned}
\int d^3 v_2 \hat{P}_{12}(\mathbf{k}, t) &= \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \int d^3 v_2 \mathcal{R}\{\hat{S}\} \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega(\omega - \omega_1 - \omega_2)} \\
&= \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \int d^3 v_2 \frac{(-i)\mathcal{R}\{\hat{S}\}}{\omega_1 + \omega_2} \left[1 - e^{-i(\omega_1 + \omega_2)t} \right].
\end{aligned}
\tag{2.105}
$$

Identifying an $\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)-1$ in the last term of $\int d^3v_2\mathcal{R}\{\hat{S}\}$, we find

$$\int d^3v_2\,\mathcal{R}\{\hat{S}\} = \tag{2.106}$$
$$-\frac{4\pi i q_1 q_2}{m_1 k_1^2}\int d^3v_2 \frac{f(\mathbf{v}_2)\,\mathbf{k}_1\cdot\partial f(\mathbf{v}_1)/\partial\mathbf{v}_1\big[(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_2)\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)-(\omega_1+\omega_2)\big(\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)-1\big)\big]}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_1)(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_2)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)}$$
$$+\frac{4\pi i q_1 q_2}{m_2 k_1^2(\omega_1-\mathbf{k}\cdot\mathbf{v}_1)}\int d^3v_2\frac{f(\mathbf{v}_1)\mathbf{k}_1\cdot\partial f(\mathbf{v}_2)/\partial\mathbf{v}_2}{\hat{\varepsilon}(-\mathbf{k}_1\omega_2)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)}.$$

For the first term in equation 2.106, the part in square brackets is

$$(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_2)\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)-(\omega_1+\omega_2)[\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)-1] = -\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)+(\omega_1+\omega_2). \tag{2.107}$$

For the second term in equation 2.106, note that

$$\frac{f(\mathbf{v}_1)}{\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)}\int d^3v_2\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_2)/\partial\mathbf{v}_2}{\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2} = \frac{-f(\mathbf{v}_1)}{\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)}\frac{m_2k_1^2}{4\pi q_1q_2}\big[\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)-1\big] \tag{2.108}$$
$$=-\frac{k_1^2m_2}{4\pi q_1q_2}\frac{f(\mathbf{v}_1)\big[\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)-1\big]}{\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)}\frac{1}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)}\left[1+\frac{4\pi q_1q_2}{m_2k_1^2}\int d^3v_2\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_2)/\partial\mathbf{v}_2}{\omega_1-\mathbf{k}_1\cdot\mathbf{v}_2}\right]$$
$$=\frac{f(\mathbf{v}_1)}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)}\left[\int d^3v_2\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_2)/\partial\mathbf{v}_2}{\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2}-\big[\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)-1\big]\int d^3v_2\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_2)/\partial\mathbf{v}_2}{\omega_1-\mathbf{k}_1\cdot\mathbf{v}_2}\right]$$
$$=\frac{f(\mathbf{v}_1)}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)}\int d^3v_2\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_2)/\partial\mathbf{v}_2\big\{\omega_1-\mathbf{k}_1\cdot\mathbf{v}_2-\big[\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)-1\big](\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)\big\}}{(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_2)},$$

and that in the last line of equation 2.108, the part in braces is equal to equation 2.107:

$$\omega_1-\mathbf{k}_1\cdot\mathbf{v}_2-\big[\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)-1\big](\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2) = -\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)+(\omega_1+\omega_2). \tag{2.109}$$

Putting the results of equations 2.107, 2.108 and 2.109 into 2.106 yields

$$\int d^3v_2\mathcal{R}\{\hat{S}\} = \frac{4\pi i q_1q_2}{k_1^2}\int d^3v_2\mathbf{k}_1\cdot\left(\frac{1}{m_1}\frac{\partial}{\partial\mathbf{v}_1}-\frac{1}{m_2}\frac{\partial}{\partial\mathbf{v}_2}\right)f(\mathbf{v}_1)f(\mathbf{v}_2) \tag{2.110}$$
$$\cdot\frac{\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)-(\omega_1+\omega_2)}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_1)(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_2)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)}.$$

When inserting equation 2.110 into 2.105, the terms with $\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)$ in the numerator vanish upon completing the ω_2 integral. Then, finally putting the result the into the collisional current of equation 2.87, we find that the collisional current can be written in the Landau form

$$\mathbf{J}_v = \int d^3v_2\mathcal{Q}(\mathbf{v}_1,\mathbf{v}_2)\cdot\left(\frac{1}{m_2}\frac{\partial}{\partial\mathbf{v}_2}-\frac{1}{m_1}\frac{\partial}{\partial\mathbf{v}_1}\right)f(\mathbf{v}_1)f(\mathbf{v}_2), \tag{2.111}$$

which has both a diffusion component (due to the $\partial/\partial\mathbf{v}_1$ term) and a drag component (due to the

$\partial/\partial\mathbf{v}_2$ term). Here $\mathcal{Q}$ is the tensor kernel

$$\begin{aligned}\mathcal{Q}(\mathbf{v}_1,\mathbf{v}_2) &= \frac{(4\pi)^2 q_1^2 q_2^2}{m_1}\int\frac{d^3k_1}{(2\pi)^3}\frac{-i\mathbf{k}_1\mathbf{k}_1}{k_1^4}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi} \\ &\cdot \frac{(1-e^{-i(\omega_1+\omega_2)t})(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_1)}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_1)(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_2)\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_1)}.\end{aligned} \tag{2.112}$$

Of the four terms in the numerator of equation 2.112, the two proportional to $\mathbf{k}_1\cdot\mathbf{v}_1$ have overall odd parity in $\mathbf{k}_1$ and vanish upon doing the $\mathbf{k}_1$ integral. The term with just ω_2 vanishes for stable plasmas and is much smaller than the exponentially growing terms for unstable plasmas. Thus, it can be neglected. The collisional kernel can then be written in the form

$$\mathcal{Q}(\mathbf{v}_1,\mathbf{v}_2) = \frac{(4\pi)^2 q_1^2 q_2^2}{m_1}\int\frac{d^3k_1}{(2\pi)^3}\frac{-i\mathbf{k}_1\mathbf{k}_1}{k_1^4}p_1(\mathbf{k}_1)p_2(\mathbf{k}_1) \tag{2.113}$$

in which p_1 and p_2 are defined by

$$p_1(\mathbf{k}_1) = \int\frac{d\omega_1}{2\pi}\frac{e^{-i\omega_1 t}}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_1)(\omega_1-\mathbf{k}_1\cdot\mathbf{v}_2)} \tag{2.114}$$

and

$$p_2(\mathbf{k}_1) = \int\frac{d\omega_2}{2\pi}\frac{\omega_2\ e^{-i\omega_2 t}}{\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_1)(\omega_2+\mathbf{k}_1\cdot\mathbf{v}_2)}. \tag{2.115}$$

Equations 2.113, 2.114 and 2.115 are identical to equations 2.34, 2.35 and 2.36 that were derived in section 2.1.3 using the test particle method. Thus, the same collision operator has been found. We carry out the inverse Laplace transforms in the same way as shown in equations 2.37 – 2.39. In doing so, we account for the poles at $\omega = \pm\mathbf{k}\cdot\mathbf{v}$, which leads to the conventional Lenard-Balescu collisional kernel, and for poles at $\hat{\varepsilon} = 0$. If instabilities are present, the poles at $\hat{\varepsilon} = 0$ produce temporally growing responses.

We make a final substitution in which we identify the species that we have labeled $f(\mathbf{v}_1)$ as species s. The species that interacts with s, which has been labeled $f(\mathbf{v}_2)$ up to now, we label s'. The species s' represent the entire plasma (including s itself) and can be split into different components (i.e., individual s'). Thus, the total s response is due to the sum of the s' components. We also drop the subscripts on $\mathbf{k}_1$ and $\mathbf{v}_1$ and label $\mathbf{v}_2$ as $\mathbf{v}'$.

After these substitutions, the final kinetic equation for species s is $\partial f_s/\partial t + \mathbf{v}\cdot\partial f_s/\partial\mathbf{x} = C(f_s) = \sum_{s'} C(f_s, f_{s'})$ in which

$$\boxed{C(f_s,f_{s'}) = -\frac{\partial}{\partial\mathbf{v}}\cdot\int d^3v'\,\mathcal{Q}\cdot\left(\frac{1}{m_{s'}}\frac{\partial}{\partial\mathbf{v}'} - \frac{1}{m_s}\frac{\partial}{\partial\mathbf{v}}\right)f_s(\mathbf{v})f_{s'}(\mathbf{v}')} \tag{2.116}$$

is the component collision operator describing collisions between species s and s' and $\mathcal{Q} = \mathcal{Q}_{\rm LB} + \mathcal{Q}_{\rm IE}$ is the collisional kernel. The collisional kernel consists of the Lenard-Balescu term

$$\boxed{\mathcal{Q}_{\rm LB} = \frac{2q_s^2 q_{s'}^2}{m_s} \int d^3k \frac{\mathbf{kk}}{k^4} \frac{\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]}{|\hat{\varepsilon}(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2}} \tag{2.117}$$

that describes the conventional Coulomb scattering of individual particles and the instability-enhanced term

$$\boxed{\mathcal{Q}_{\rm IE} = \frac{2q_s^2 q_{s'}^2}{\pi m_s} \int d^3k \frac{\mathbf{kk}}{k^4} \sum_j \frac{\gamma_j \, \exp(2\gamma_j t)}{[(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v})^2 + \gamma_j^2][(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v}')^2 + \gamma_j^2]|\partial\hat{\varepsilon}(\mathbf{k},\omega)/\partial\omega|^2_{\omega_j}}} \tag{2.118}$$

that describes the scattering of particles by collective fluctuations. We can also write the dielectric function in the familiar form

$$\hat{\varepsilon}(\mathbf{k},\omega) = 1 + \sum_{s'} \frac{4\pi q_{s'}^2}{k^2 m_{s'}} \int d^3v \frac{\mathbf{k}\cdot\partial f_{s'}(\mathbf{v})/\partial\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}}. \tag{2.119}$$

We have used the notation $\omega_j = \omega_{R,j} + i\gamma_j$ where $\omega_{R,j}$ and γ_j are the real and imaginary parts of the $j^{\rm th}$ root of the dielectric function equation 2.119. Equations 2.116, 2.117 and 2.118 provide a BBGKY hierarchy derivation of the same collision operator that was derived in section 2.1 (equations 2.43, 2.44 and 2.45) using a discrete particle approach.

2.3 Total Versus Component Collision Operators

The total collision operator $C(f_s)$ for the evolution equation of species s is a sum of the collision operators describing collisions between s and each species s' (including itself, $s' = s$); thus $C(f_s) = \sum_{s'} C(f_s, f_{s'})$. This total collision operator appears from equations 2.43, 2.44 and 2.45 (or, equivalently 2.116, 2.117 and 2.118) to have four terms: terms for "drag" and "diffusion" (from the $\partial/\partial\mathbf{v}'$ and $\partial/\partial\mathbf{v}$ derivatives respectively) using both the Lenard-Balescu collisional kernel of equation 2.43, and the instability-enhanced collisional kernel of equation 2.44. However, there are actually only three non-zero terms because the total instability-enhanced contribution to drag vanishes. To show this, we write the total instability-enhanced collision operator as

$$C_{\rm IE}(f_s) = -\frac{\partial}{\partial\mathbf{v}} \cdot \sum_{s'} \int d^3v' \mathcal{Q}_{\rm IE} \cdot \left(\frac{1}{m_{s'}} \frac{\partial}{\partial\mathbf{v}'} - \frac{1}{m_s} \frac{\partial}{\partial\mathbf{v}} \right) f_s(\mathbf{v}) f_{s'}(\mathbf{v}') \tag{2.120}$$

$$= \frac{\partial}{\partial\mathbf{v}} \cdot \left[\mathcal{D}_{\rm IE,diff} \cdot \frac{\partial f_s(\mathbf{v})}{\partial\mathbf{v}} \right] - \frac{\partial}{\partial\mathbf{v}} \cdot \left[\mathbf{D}_{\rm IE,drag} f_s(\mathbf{v}) \right] \tag{2.121}$$

in which

$$\mathcal{D}_{\mathrm{IE,diff}} \equiv \sum_{s'} \int d^3v' \mathcal{Q}_{\mathrm{IE}} \frac{f_{s'}(\mathbf{v}')}{m_s} \tag{2.122}$$

and

$$\mathbf{D}_{\mathrm{IE,drag}} \equiv \sum_{s'} \int d^3v' \mathcal{Q}_{\mathrm{IE}} \cdot \frac{1}{m_{s'}} \frac{\partial f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'}. \tag{2.123}$$

Evaluating the dielectric function, equation 2.18, at it's roots (ω_j) and multiplying by $(\omega_j - \mathbf{k}\cdot\mathbf{v})^*/(\omega_j - \mathbf{k}\cdot\mathbf{v})^*$ inside the integral gives

$$\hat{\varepsilon}(\mathbf{k},\omega_j) = 1 + \sum_{s'} \frac{4\pi q_{s'}^2}{k^2 m_{s'}} \int d^3v' \frac{(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v}' - i\gamma_j)\mathbf{k}\cdot\partial f_{s'}/\partial \mathbf{v}'}{(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v}')^2 + \gamma_j^2}. \tag{2.124}$$

The real and imaginary parts of equation 2.124 individually vanish; $\Re\{\hat{\varepsilon}(\mathbf{k},\omega_j)\} = \Im\{\hat{\varepsilon}(\mathbf{k},\omega_j)\} = 0$. The imaginary part is

$$\Im\{\hat{\varepsilon}(\mathbf{k},\omega_j)\} = \sum_{s'} \frac{4\pi q_{s'}^2}{k^2 m_{s'}} \int d^3v' \frac{(-\gamma_j)\mathbf{k}\cdot\partial f_{s'}(\mathbf{v}')/\partial \mathbf{v}'}{(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v}')^2 + \gamma_j^2}. \tag{2.125}$$

Equation 2.125 shows that the term proportional to $\omega_{R,j}$ in equation 2.124 is zero; thus the real part of equation 2.124 can be written

$$\Re\{\hat{\varepsilon}(\mathbf{k},\omega_j)\} = 1 - \sum_{s'} \frac{4\pi q_{s'}^2}{k^2 m_{s'}} \int d^3v' \frac{\mathbf{k}\cdot\mathbf{v}'\,\mathbf{k}\cdot\partial f_{s'}(\mathbf{v}')/\partial \mathbf{v}'}{(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v}')^2 + \gamma_j^2}. \tag{2.126}$$

Putting equation 2.44 into $\mathbf{D}_{\mathrm{IE,drag}}$ in equation 2.123, we find $\mathbf{D}_{\mathrm{IE,drag}}(f_s) \propto \Im\{\hat{\varepsilon}(\mathbf{k},\omega_j)\} = 0$:

$$\mathbf{D}_{\mathrm{IE,drag}}(f_s) = -\frac{q_s^2}{2\pi^2 m_s} \int d^3k \frac{\mathbf{k}}{k^2} \frac{\exp(2\gamma_j t)}{[(\omega_{R,j}^2 - \mathbf{k}\cdot\mathbf{v})^2 + \gamma_j^2]\left|\partial\hat{\varepsilon}(\mathbf{k},\omega)/\partial\omega\right|_{\omega_j}^2} \Im\{\hat{\varepsilon}(\mathbf{k},\omega_j)\} = 0. \tag{2.127}$$

Thus, only a diffusion term survives in the instability-enhanced portion of the total collision operator $C_{\mathrm{IE}}(f_s)$.

Although the instability-enhanced drag is zero in the total collision operator $C(f_s)$, it is not necessarily zero in each component collision operator $C(f_s, f_{s'})$. Identifying the component collision operators in the form of equation 2.45 is particularly useful because they do not each cause evolution of f_s on similar time scales. For example, like-particle collisions ($s = s'$) tend to dominate unlike-particle collisions for short times. The time scale for which unlike-particle collisions matter may be longer than those of interest, or those of mechanisms external to the plasma theory such as neutral collisions or losses to boundaries. Thus, although the instability-enhanced contribution to drag in the total collision operator vanishes, it remains a useful term in describing the individual collision operator components.

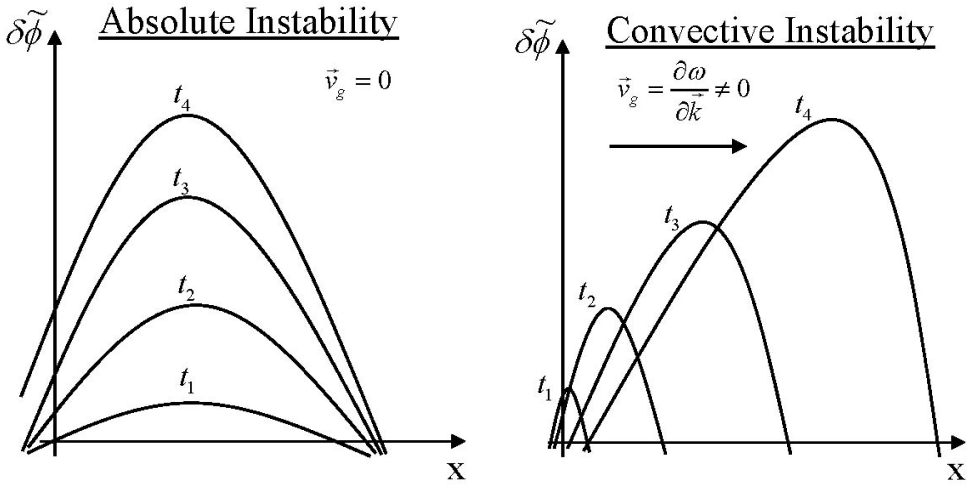

Figure 2.1: Illustration of how (a) absolute instabilities grow in time at a fixed spatial locations and how (b) convective instabilities move as they grow so that fluctuations grow in the reference frame of the wave.

2.4 Interpretation of $e^{2\gamma t}$

A proper interpretation of time, t, in equation 2.44 requires consideration of the nature of instabilities present in the plasma. If the instabilities are absolute, i.e., the modes grow continually in time at a fixed spatial location with a vanishing group velocity, one can consider the collision operator at a fixed location to be dependent on time and hence f_s would evolve in time at that location. For example, if an absolute instability were to be turned on at some time t_o, the time t in equation 2.44 would simply refer to the progression of time, at some location, after the instability is present. In this case, the above analysis will hold only for a few growth times, $\tau \sim 1/\gamma$, before nonlinear effects become important. The linear theory, however, would be valid for the short time scale evolution of plasmas with absolute instabilities.

Convective modes, on the other hand, propagate through the plasma with a finite group velocity $\mathbf{v}_g$. For these instabilities the fluctuation level at a fixed location in space does not grow or decay in time; rather, the waves grow as they propagate and thus the fluctuation amplitude changes for different spatial locations. See the illustration in figure 2.1. In this case, time t in equation 2.44 is the time it takes a growing mode to travel from its origin to the spatial location of interest. For convective modes,

f_s does not evolve in time at a fixed spatial location; rather, it evolves in space along the direction of propagation of the convective modes. Time is interpreted in a frame of reference moving with the group velocity of the wave to yield

$$2\gamma t = 2\int_{\mathbf{x}_o(\mathbf{k})}^{\mathbf{x}} d\mathbf{x}' \cdot \frac{\mathbf{v}_g \gamma}{|\mathbf{v}_g|^2} \tag{2.128}$$

in which $\mathbf{x}_o(\mathbf{k})$ is the location where the mode with wavenumber $\mathbf{k}$ becomes unstable, $\mathbf{x}$ is the measurement location, and $\mathbf{x}'$ is the path between $\mathbf{x}_o$ and $\mathbf{x}$ that the mode with wavenumber $\mathbf{k}$ follows.

For convective instabilities in a homogeneous finite domain, the same $\mathbf{k}$ are unstable throughout the region since the plasma dielectric function is uniform. In this case the coordinate system can be chosen such that $\mathbf{x}_o(\mathbf{k}) = 0$. However, if small spatial inhomogeneities are present in the equilibrium, or if scattering by either the convective instabilities or Coulomb interactions alters the plasma dielectric, different $\mathbf{k}$ may be unstable at different locations in the domain and care has to be taken when determining the spatial integration limits.

2.5 Validity of the Kinetic Theory

To estimate the domain length over which instabilities can grow (either in time for absolute instabilities or space for convective instabilities) before nonlinear effects become important we need to consider equation 2.13. A conservative estimate for the maximum domain length can be obtained by considering just a single term from its left side, which implies the requirement $|\delta f|/f \lesssim 1$. Equations 2.20 and 2.22 lead to the scaling relationship

$$\delta\hat{f} \sim \frac{q}{m}\frac{\mathbf{k}\cdot\partial f/\partial\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}}\delta\hat{\phi}. \tag{2.129}$$

Typically for electrons, $\omega - \mathbf{k}\cdot\mathbf{v} \sim kv_{Te}$ for a characteristic thermal electron speed v_{Te}. Using this, one finds the validity condition for the linear model reduces to the intuitive requirement that the Coulombic potential energy level of the fluctuations cannot exceed the ambient thermal energy of the plasma, $q\delta\phi/T_e \lesssim 1$. The more general condition is

$$\frac{4\pi e^2}{m}\frac{1}{f}\frac{\partial f}{\partial \mathbf{v}} \cdot \int d^3k\ \frac{\mathbf{k}}{k^2}\left[\frac{\delta\big[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')\big]}{\hat{\varepsilon}(\mathbf{k},\mathbf{k}\cdot\mathbf{v})} + \sum_j \frac{e^{\gamma_j t}}{(\omega_j - \mathbf{k}\cdot\mathbf{v})(\omega_j - \mathbf{k}\cdot\mathbf{v}')\partial\hat{\varepsilon}(\mathbf{k},\omega)/\partial\omega\big|_{\omega_j}}\right] \lesssim 1. \tag{2.130}$$

It is difficult to extract any more information from equation 2.130 without specifying the nature of particular instabilities. To check that equation 2.130 is consistent with that of previous models when

the the plasma is stable, consider a typical illustrative example of the Lenard-Balescu equation when Debye shielding is included, $\hat{\varepsilon} = 1 + 1/k^2\lambda_{De}^2$. In this case, equation 2.130 reduces to $\ln\Lambda/n\lambda_{De}^3 \lesssim 1$, which is consistent with previous analysis which used the BBGKY hierarchy to justify the derivation of the Lenard-Balescu equation using the test particle method [57].

Chapter 3

Properties of $C(f_s)$ and Comparison to Quasilinear Theory

Since its introduction in the early 1960's, "quasilinear theory" has become an umbrella term for a host of different theories. For example, it has been applied in theories describing the interaction of particles and applied waves [67], to anomalous particle transport in fusion plasmas [68] and to the diffusion of magnetic field in fusion devices [68]. Here we use the term "conventional quasilinear theory" to refer to the original theories of Vedenov, Velikhov and Sagdeev [8, 9, 36], Drummond and Pines [10], and Bernstein and Englemann [35], which used the collisionless Vlasov equation to develop an effective "collision operator" that describes wave-particle scattering due to plasma fluctuations. In conventional quasilinear theory, the origin of fluctuations is not specified and the final effective collision operator requires input of an initial fluctuation level (or spectral energy density) that must be determined external to the theory. The effective collision operator yields a diffusion equation.

In this chapter, we show that the instability-enhanced term of the total collision operator for species s, which is a sum of the component collision operators describing collisions of s with each species s' $C(f_s) = \sum_{s'} C(f_s, f_{s'})$, fits into the diffusion equation framework of quasilinear theory. The instability-enhanced contribution to the total collision operator may be considered an extension of conventional quasilinear theory for the case that instabilities arise within the plasma. This is because the kinetic prescription determines the spectral energy density by self-consistently accounting for the continuing source of fluctuations from discrete particle motion. This leads to important extensions of the physical properties of the resultant operator such as conservation laws between individual species, that the Boltzmann $\mathcal{H}$-theorem is satisfied, and that internal instabilities drive the individual species distribution functions toward Maxwellians. We provide proofs of these properties, and others, in section 3.4.

3.1 Conventional Quasilinear Theory

Conventional quasilinear theory can be derived in a manner similar to the test-particle approach that was used in chapter 2 to derive a kinetic theory. However, it is a "collisionless" theory because it is based on the Vlasov equation [34]

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s} \mathbf{E} \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0 \tag{3.1}$$

in which f_s is the same "ensemble averaged" distribution function that we used in chapter 2. We again assume a magnetic-field-free plasma: $\mathbf{B} = 0$. The Vlasov equation 3.1 is the plasma kinetic equation neglecting the collision operator.

The conventional derivation of quasilinear theory proceeds by separating spatially-smoothed and fluctuation-scale components of f_s: $f_s = f_{so} + f_{s1}$. The smoothed components are defined by the volume average

$$f_{so}(\bar{\mathbf{x}}, \mathbf{v}, t) \equiv \frac{1}{V} \int d^3x \, f_s = \langle f_s \rangle. \tag{3.2}$$

It is assumed that $\langle f_{s1} \rangle = 0$. The V in equation 3.2 refers to a macroscopic volume over which the d^3x integral of the spatial average is taken. The quasilinear theory equations describe the evolution of f_{so}; thus it assumes that the plasma is uniform on the spatial scales characteristic of the volume average. The smoothed distribution f_{so} can only vary on spatial scales larger than the characteristic scale of V and this spatial scale is denoted $\bar{\mathbf{x}}$.

Appendix B shows that the effect of equilibrium electrostatic fields affect the collision operator (or "effective" collision operator in this case) only when the electrostatic potential satisfies

$$\frac{1}{k\phi_o} \frac{\partial \phi_o}{\partial \mathbf{x}} \gtrsim 1, \tag{3.3}$$

which implies that the electric field is strong and, as a consequence, that the plasma is not quasineutral. We assume only weak fields are present and thus take $\mathbf{E}_o = 0$. Applying the assumptions $\langle \mathbf{E}_1 \rangle = 0$, $\langle f_{s1} \rangle = 0$ and spatially averaging equation 3.1 (using equation 3.2) yields

$$\frac{\partial f_{so}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{so}}{\partial \bar{\mathbf{x}}} = -\frac{q_s}{m_s} \frac{\partial}{\partial \mathbf{v}} \cdot \langle \mathbf{E}_1 \, f_{s1} \rangle. \tag{3.4}$$

An equation for f_{s1} is obtained by putting $f_s = f_{so} + f_{s1}$ into equation 3.1, then subtracting equation 3.4

from the result. With this, we find

$$\frac{\partial f_{s1}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{s1}}{\partial \mathbf{x}} = -\frac{q_s}{m_s}\left(\mathbf{E}_1 \cdot \frac{\partial f_{so}}{\partial \mathbf{v}} + \mathbf{E}_1 \cdot \frac{\partial f_{s1}}{\partial \mathbf{v}} - \left\langle \mathbf{E}_1 \cdot \frac{\partial f_{s1}}{\partial \mathbf{v}} \right\rangle\right). \tag{3.5}$$

Just as we did in the test particle method in chapter 2.1, we assume that the quasilinear approximation

$$\left|\mathbf{E}_1 \cdot \frac{\partial f_{s1}}{\partial \mathbf{v}} - \left\langle \mathbf{E}_1 \cdot \frac{\partial f_{s1}}{\partial \mathbf{v}} \right\rangle\right| \ll \left|\mathbf{E}_1 \cdot \frac{\partial f_{so}}{\partial \mathbf{v}}\right| \tag{3.6}$$

holds. With this assumption we neglect nonlinear wave effects such as mode coupling, nonlinear saturation, and turbulence. Like the kinetic theory of chapter 2, quasilinear theory is limited to describing the effects of instabilities in a linear growth regime. Dropping the nonlinear terms in equation 3.5 yields

$$\frac{\partial f_{s1}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{s1}}{\partial \mathbf{x}} = -\frac{q_s}{m_s}\mathbf{E}_1 \cdot \frac{\partial f_{so}}{\partial \mathbf{v}}. \tag{3.7}$$

The original quasilinear references [8–10] solve equation 3.7 by assuming that the time dependence of f_{s1} and $\mathbf{E}_1$ obey

$$f_{s1}(\mathbf{x}, \mathbf{v}, t) = \sum_j \bar{f}_{s1}(\mathbf{x}, \mathbf{v}) e^{-i\omega_j t} \quad \text{and} \quad \mathbf{E}_1 = \sum_j \bar{\mathbf{E}}(\mathbf{x}) e^{-i\omega_j t} \tag{3.8}$$

in which $\omega_j(\mathbf{k})$ is the dispersion relation, i.e., the roots of $\hat{\varepsilon} = 0$.

It is, perhaps, beneficial at this point to compare this foundation for quasilinear theory to what we used in developing the instability-enhanced collision operator. The quasilinear theory starting point is the Vlasov equation, which describes the evolution of the ensemble averaged distribution f_s, whereas the linear kinetic theory started from the Klimontovich equation, which describes the evolution of the more fundamental discrete particle distribution F. So at first, it seems that the theories must describe the evolution of quite different quantities. However, comparing equations 2.9 and 2.11 to equations 3.4 and 3.7, shows that formally the equations are quite similar. In fact, if $f_{so} \to f_s$, $f_{s1} \to \delta f$, $\mathbf{E}_1 \to \delta\mathbf{E}$, and the spatial average of equation 3.2 were replaced with the ensemble average of equation 2.25, the two sets of equations would be precisely the same. The neglect of nonlinear terms in equations 2.11 and 3.5 also suggest that the two theories are confined to similar fluctuation levels. However, quasilinear theory does not account for discrete particle effects and results of the two theories will be different. Also, quasilinear theory does not use any of Maxwell's equations to relate density fluctuations to field fluctuations. In the kinetic approach, this connection was made through Gauss's law (equation 2.12).

For this reason, the plasma dielectric function does not naturally come into the quasilinear theory, and the ω_j are imposed from an external determination of the dielectric function.

Using the assumption of equation 3.8 and Fourier transforming, equation 3.7 can be written

$$\hat{f}_{s1}(\mathbf{k},\mathbf{v}) = -i\frac{q_s}{m_s}\sum_j \frac{\hat{\mathbf{E}}_1(\mathbf{k})\cdot\partial f_{so}/\partial\mathbf{v}}{\omega_j - \mathbf{k}\cdot\mathbf{v}} \tag{3.9}$$

in which the "hat" denotes Fourier transformed variables. The right side of equation 3.4 is

$$\begin{aligned}\langle \mathbf{E}_1 f_{s1}\rangle &= \frac{1}{V}\int d^3x\, \mathbf{E}_1 f_{s1} = \frac{1}{V}\int d^3x \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}}\hat{f}_{s1}(\mathbf{k}) \int \frac{d^3k'}{(2\pi)^3} e^{-i\mathbf{k}'\cdot\mathbf{x}}\hat{\mathbf{E}}_1(\mathbf{k}') \\ &= \frac{1}{(2\pi)^3 V}\int d^3k \hat{f}_{s1}(\mathbf{k}) \int d^3k' \hat{\mathbf{E}}_1(\mathbf{k}')\delta(\mathbf{k}+\mathbf{k}') = \frac{1}{(2\pi)^3 V}\int d^3k\, \hat{f}_{s1}(\mathbf{k})\, \hat{\mathbf{E}}_1(-\mathbf{k}).\end{aligned} \tag{3.10}$$

Inserting equation 3.9 into equation 3.10 gives

$$\langle \mathbf{E}_1 f_{s1}\rangle = \frac{1}{(2\pi)^3 V}\frac{q_s}{m_s}\sum_j \int d^3k \frac{\hat{\mathbf{E}}(-\mathbf{k})\hat{\mathbf{E}}(\mathbf{k})\cdot\partial f_{so}/\partial\mathbf{v}}{i(\omega_j - \mathbf{k}\cdot\mathbf{v})}. \tag{3.11}$$

Since $\mathbf{E}_1$ is assumed to be electrostatic fluctuations, it can be written in terms of a potential, $\hat{\mathbf{E}}_1(\mathbf{k}) = -i\mathbf{k}\phi(\mathbf{k})$. Thus, $\hat{\mathbf{E}}_1(\mathbf{k})\hat{\mathbf{E}}_1(-\mathbf{k}) = \hat{\mathbf{E}}_1(\mathbf{k})\cdot\hat{\mathbf{E}}_1(-\mathbf{k})\mathbf{k}\mathbf{k}/k^2$. We next define the spectral energy density as

$$\boxed{\mathcal{E}^{\mathrm{ql}}(\mathbf{k}) \equiv \frac{1}{(2\pi)^3 V}\frac{\hat{\mathbf{E}}_1(\mathbf{k})\cdot\hat{\mathbf{E}}_1(-\mathbf{k})}{8\pi} = \sum_j \frac{|E_1(\mathbf{k},t=0)|^2}{(2\pi)^3 V}\frac{e^{2\gamma_j t}}{8\pi},} \tag{3.12}$$

in which the last step follows from equation 3.8 and the reality condition $\omega(\mathbf{k}) = -\omega^*(-\mathbf{k})$. Putting equation 3.11 into equation 3.4 yields the quasilinear diffusion equation

$$\boxed{\frac{\partial f_{so}}{\partial t} + \mathbf{v}\cdot\frac{\partial f_{so}}{\partial\bar{\mathbf{x}}} = \frac{\partial}{\partial\mathbf{v}}\cdot\mathcal{D}_v\cdot\frac{\partial f_{so}}{\partial\mathbf{v}}} \tag{3.13}$$

in which the quasilinear diffusion tensor is

$$\mathcal{D}_v \equiv \frac{q_s^2}{m_s^2}8\pi\sum_j\int d^3k\,\frac{\mathbf{k}\mathbf{k}}{k^2}\frac{i\,\mathcal{E}_j^{\mathrm{ql}}(\mathbf{k})}{\omega_j - \mathbf{k}\cdot\mathbf{v}}. \tag{3.14}$$

The diffusion tensor is often simplified by multiplying equation 3.14 by $(\omega_j^* - \mathbf{k}\cdot\mathbf{v})/(\omega_j^* - \mathbf{k}\cdot\mathbf{v})$ to give

$$\mathcal{D}_v = 8\pi\frac{q_s^2}{m_s^2}\sum_j\int d^3k\frac{\mathbf{k}\mathbf{k}}{k^2}\frac{i\,\mathcal{E}_j^{\mathrm{ql}}(\mathbf{k})}{\left[(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v})^2 + \gamma^2\right]}\left(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v} - i\gamma\right) \tag{3.15}$$

in which the integrals with $\omega_{R,j} - \mathbf{k}\cdot\mathbf{v}$ will vanish because they have odd parity in $\mathbf{k}$ [in which we use the property $\omega_{R,j}(-\mathbf{k}) = -\omega_{R,j}(\mathbf{k})$]. This leaves

$$\boxed{\mathcal{D}_v = \frac{q_s^2}{m_s^2}8\pi\sum_j\int d^3k\frac{\mathbf{k}\mathbf{k}}{k^2}\frac{\gamma_j\,\mathcal{E}_j^{\mathrm{ql}}(\mathbf{k})}{[(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v})^2 + \gamma_j^2]}.} \tag{3.16}$$

For $\gamma \ll \omega_{R,j}$, the Lorentzian in equation 3.16 is approximately a δ-function (recall equation 2.46), and the diffusion tensor has the approximate form

$$\mathcal{D}_v \approx 8\pi^2 \frac{q_s^2}{m_s^2} \sum_j \int d^3k \frac{\mathbf{kk}}{k^2} \delta\big(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v}\big) \mathcal{E}_j^{\mathrm{ql}}(\mathbf{k}). \tag{3.17}$$

Equations 3.13 and 3.16 or 3.17 constitute quasilinear theory.

3.2 Davidson's Quasilinear Theory

In section 2.2, a collision operator was derived using the BBGKY hierarchy. During the analysis, specifically at equation 2.90, contributions arose from both a source term ($\hat{S}$), due to discrete particle fluctuations, and from an initial pair correlation term [$\tilde{P}_{12}(t=0)$]. According to the BBGKY hierarchy ordering, the pair correlation term is higher order than the source term; thus it was neglected in comparison to the source term. However, Davidson [58] has considered taking only the initial pair correlation as a way to derive a quasilinear theory from the BBGKY hierarchy. Like conventional quasilinear theory, it is "collisionless" since it neglects the discrete particle source term. In this section, we will review Davidson's approach and give a brief comparison of his result to that of the conventional quasilinear theory.

Dropping the source term, S, in equation 2.90 and only considering the initial pair correlation, $\tilde{P}_{12}(t=0)$, leaves

$$\hat{P}_{1,2}(\mathbf{k}_1, \mathbf{v}_1, \mathbf{v}_2, \omega) = \frac{\tilde{P}_{1,2}(\mathbf{k}_1, \mathbf{v}_1, \mathbf{v}_2, t=0)}{-i\omega + L_1(\mathbf{k}_1) + L_2(-\mathbf{k}_1)}. \tag{3.18}$$

Substituting equation 2.92 for $1/[-i\omega + L_1(\mathbf{k}_1) + L_2(-\mathbf{k}_1)]$, we find

$$\hat{P}_{1,2}(\mathbf{k}_1, \mathbf{v}_1, \mathbf{v}_2, \omega) = \frac{1}{(2\pi)^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} d\omega_1 d\omega_2 \frac{\mathcal{R}\big\{\tilde{P}_{12}(t=0)\big\}}{-i(\omega - \omega_1 - \omega_2)}. \tag{3.19}$$

Inverting the Laplace transform yields

$$\hat{P}_{1,2}(\mathbf{k}_1, \mathbf{v}_1, \mathbf{v}_2, t) = \frac{-1}{(2\pi)^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} d\omega_1 d\omega_2 \, e^{-i(\omega_1+\omega_2)t} \mathcal{R}\big\{\tilde{P}_{1,2}(\mathbf{k}_1, \mathbf{v}_1, \mathbf{v}_2, t=0)\big\}. \tag{3.20}$$

Applying the Frieman-Rutherford operator from equations 2.93 and 2.94, gives

$$\hat{P}_{1,2} = -\underbrace{\left[\int \frac{d\omega_1}{2\pi} e^{-i\omega_1 t}\, \frac{i}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_1}\left\{1 - \frac{4\pi q_1 q_2}{m_1 k_1^2}\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_1)/\partial\mathbf{v}_1}{\hat{\varepsilon}(\mathbf{k}_1,\omega_1)}\int \frac{d^3v_1}{\omega_1 - \mathbf{k}_1\cdot\mathbf{v}_1}\right\}\right]}_{\equiv h_1} \tag{3.21}$$
$$\times \underbrace{\left[\int \frac{d\omega_2}{2\pi} e^{-i\omega_2 t}\, \frac{i}{\omega_2 + \mathbf{k}_1\cdot\mathbf{v}_2}\left\{1 + \frac{4\pi q_1 q_2}{m_2 k_1^2}\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_2)/\partial\mathbf{v}_2}{\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)}\int \frac{d^3v_2}{\omega_2 + \mathbf{k}_1\cdot\mathbf{v}_2}\right\}\right]}_{\equiv h_2} \tilde{P}_{1,2}(\mathbf{k}_1,\mathbf{v}_1,\mathbf{v}_2,t=0).$$

Evaluating the ω_1 integral to calculate h_1, yields

$$h_1 = -e^{-i\mathbf{k}_1\cdot\mathbf{v}_1 t} + \frac{4\pi q_1 q_2}{m_1 k_1^2}\mathbf{k}_1\cdot\frac{\partial f(\mathbf{v}_1)}{\partial\mathbf{v}_1}\int d^3v_1'\Bigg[\frac{e^{-i\mathbf{k}_1\cdot\mathbf{v}_1 t}}{\mathbf{k}_1\cdot(\mathbf{v}_1 - \mathbf{v}_1')\hat{\varepsilon}(\mathbf{k}_1,\mathbf{k}_1\cdot\mathbf{v}_1)} \tag{3.22}$$
$$- \frac{e^{-i\mathbf{k}_1\cdot\mathbf{v}_1' t}}{\mathbf{k}_1\cdot(\mathbf{v}_1 - \mathbf{v}_1')\hat{\varepsilon}(\mathbf{k}_1,\mathbf{k}_1\cdot\mathbf{v}_1')} + \sum_j \frac{e^{-i\omega_j t}}{(\omega_j - \mathbf{k}_1\cdot\mathbf{v}_1)(\omega_j - \mathbf{k}_1\cdot\mathbf{v}_1')[\partial\hat{\varepsilon}(\mathbf{k}_1,\omega_1)/\partial\omega_1]_{\omega_j}}\Bigg].$$

The h_2 term can be evaluated analogously. It is the same as the h_1 term except that $\mathbf{v}_1 \to \mathbf{v}_2$, $\mathbf{k}_1 \to -\mathbf{k}_1$, and $\omega_j \to \omega_j'$ where ω_j' solves $\hat{\varepsilon}(-\mathbf{k}_1,\omega_2) = 0$ and obeys the relation $\omega_j' = -\omega_j^*$. This yields

$$h_2 = -e^{i\mathbf{k}_1\cdot\mathbf{v}_2 t} - \frac{4\pi q_1 q_2}{m_2 k_1^2}\mathbf{k}_1\cdot\frac{\partial f(\mathbf{v}_2)}{\partial\mathbf{v}_2}\int d^3v_2'\Bigg[-\frac{e^{i\mathbf{k}_1\cdot\mathbf{v}_2 t}}{\mathbf{k}_1\cdot(\mathbf{v}_2 - \mathbf{v}_2')\hat{\varepsilon}(-\mathbf{k}_1,-\mathbf{k}_1\cdot\mathbf{v}_2)} \tag{3.23}$$
$$+ \frac{e^{i\mathbf{k}_1\cdot\mathbf{v}_2' t}}{\mathbf{k}_1\cdot(\mathbf{v}_2 - \mathbf{v}_2')\hat{\varepsilon}(-\mathbf{k}_1,-\mathbf{k}_1\cdot\mathbf{v}_2')} + \sum_j \frac{e^{-i\omega_j' t}}{(\omega_j' + \mathbf{k}_1\cdot\mathbf{v}_2)(\omega_j' + \mathbf{k}_1\cdot\mathbf{v}_2')[\partial\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)/\partial\omega_2]_{\omega_j'}}\Bigg].$$

In Davidson's derivation [58], he states that the rapidly oscillating terms proportional to $e^{i\mathbf{k}\cdot\mathbf{v}t}$ can be neglected because they phase-mix to a negligible level upon the appropriate $\mathbf{k}_1$ and velocity space integrals. In the kinetic theory of chapter 2, we were interested in a stable plasma collision operator in addition to the possibly strongly growing term. Here we keep only the strongest growing terms [i.e., those $\propto \exp(-i\omega_j t)$] of equations 3.22 and 3.23, which gives

$$\hat{P}_{1,2}(\mathbf{k}_1,\mathbf{v}_1,\mathbf{v}_2,t) = \sum_j\left(\frac{4\pi q_1 q_2}{m_1 k_1^2}\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_1)/\partial\mathbf{v}_1}{(\omega_j - \mathbf{k}_1\cdot\mathbf{v}_1)}\right)\left(\frac{4\pi q_1 q_2}{m_2 k_1^2}\frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_2)/\partial\mathbf{v}_2}{(\omega_j' + \mathbf{k}\cdot\mathbf{v}_2)}\right) \tag{3.24}$$
$$\times \int d^3v_1'\, d^3v_2' \frac{\exp[-i(\omega_j + \omega_j')t]\,\tilde{P}_{1,2}(\mathbf{k}_1,\mathbf{v}_1',\mathbf{v}_2',t=0)}{(\omega_j - \mathbf{k}_1\cdot\mathbf{v}_1')(\omega_j' + \mathbf{k}_1\cdot\mathbf{v}_2)\left[\partial\hat{\varepsilon}(\mathbf{k}_1,\omega_1)/\partial\omega_1\right]_{\omega_j}\left[\partial\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)/\partial\omega_2\right]_{\omega_j'}}.$$

Recall from equations 2.85 and 2.87 that the kinetic equation is

$$\frac{\partial f}{\partial t} + \mathbf{v}_1\cdot\frac{\partial f}{\partial\bar{\mathbf{x}}} = -\frac{\partial}{\partial\mathbf{v}_1}\cdot\frac{4\pi q_1 q_2}{m_1}\int\frac{d^3k_1}{(2\pi)^3}\frac{-i\mathbf{k}_1}{k_1^2}\int d^3v_2\,\hat{P}_{1,2}(\mathbf{k}_1,\mathbf{v}_1,\mathbf{v}_2,t). \tag{3.25}$$

Thus, we are interested in $\int d^3v_2\,\hat{P}_{12}$. Noticing that

$$\sum_j \frac{4\pi q_1 q_2}{m_2 k_1^2}\int d^3v_2 \frac{\mathbf{k}_1\cdot\partial f(\mathbf{v}_2)/\partial\mathbf{v}_2}{\omega_j' + \mathbf{k}_1\cdot\mathbf{v}_2} = \hat{\varepsilon}(-\mathbf{k}_1,\omega_j') + 1 = 1, \tag{3.26}$$

where by definition $\hat{\varepsilon}(-\mathbf{k},\omega_j')=0$. Thus, the d^3v_2 integral of equation 3.24 results in the second term in parenthesis becoming 1. We also recall from the definition of $\hat{\varepsilon}$ that

$$\left.\frac{\partial\hat{\varepsilon}(\mathbf{k}_1,\omega_1)}{\partial\omega_1}\right|_{\omega_j}\left.\frac{\partial\hat{\varepsilon}(-\mathbf{k}_1,\omega_2)}{\partial\omega_2}\right|_{\omega_j'}=-\left|\frac{\partial\hat{\varepsilon}(\mathbf{k}_1,\omega)}{\partial\omega}\right|^2_{\omega_j} \tag{3.27}$$

and that $\omega_j+\omega_j'=2i\gamma_j$. With these, we find that the evolution equation is

$$\frac{df}{dt}=-\frac{\partial}{\partial\mathbf{v}}\cdot\frac{(4\pi)^2q_1^2q_2^2}{m_1^2(2\pi)^3}\sum_j\int d^3k_1\frac{\mathbf{k}_1\mathbf{k}_1/k_1^4}{\omega_j-\mathbf{k}_1\cdot\mathbf{v}_1}\int\frac{d^3v_1'd^3v_2'\,e^{2\gamma_jt}\,i\tilde{P}_{1,2}(\mathbf{k}_1,\mathbf{v}_1',\mathbf{v}_2',t=0)}{(\omega_j-\mathbf{k}_1\cdot\mathbf{v}_1')(\omega_j'+\mathbf{k}_1\cdot\mathbf{v}_2')\left|\partial\hat{\varepsilon}(\mathbf{k}_1,\omega)/\partial\omega\right|^2_{\omega_j}}. \tag{3.28}$$

Finally, as in section 2.2.3, we again identify the species that we have labeled $f(\mathbf{v}_1)$ as species s. The species that interacts with s, which has been labeled $f(\mathbf{v}_2)$ up to now, we label s'. The species s' represents the entire plasma (including s itself) and can be split into different components (i.e., individual s'). Thus, the total s response is due to the sum of the s' components. We also drop the subscripts on $\mathbf{k}_1$ and $\mathbf{v}_1$ and label $\mathbf{v}_2$ as $\mathbf{v}'$.

After these substitutions, we find that Davidson's quasilinear theory is a diffusion equation of the same form as equation 3.13

$$\frac{\partial f_s}{\partial t}+\mathbf{v}\cdot\frac{\partial f_s}{\partial\bar{\mathbf{x}}}=\frac{\partial}{\partial\mathbf{v}}\cdot\mathcal{D}_v\cdot\frac{\partial f_s}{\partial\mathbf{v}} \tag{3.29}$$

with the same diffusion tensor

$$\mathcal{D}_v\equiv\frac{q_s^2}{m_s^2}8\pi\sum_j\int d^3k\,\frac{\mathbf{kk}}{k^2}\frac{i\mathcal{E}_j^{\mathrm{dv}}(\mathbf{k})}{\omega_j-\mathbf{k}\cdot\mathbf{v}}, \tag{3.30}$$

but where the spectral energy density is redefined as

$$\boxed{\mathcal{E}_j^{\mathrm{dv}}(\mathbf{k})=\sum_{s'}\frac{q_{s'}^2}{4\pi^2k^2\left|\partial\hat{\varepsilon}(\mathbf{k},\omega)/\partial\omega\right|^2_{\omega_j}}\int d^3v\int d^3v'\frac{\tilde{P}(\mathbf{k},\mathbf{v},\mathbf{v}',t=0)e^{2\gamma_jt}}{(\omega_j-\mathbf{k}\cdot\mathbf{v})(\omega_j^*-\mathbf{k}\cdot\mathbf{v}')}.} \tag{3.31}$$

Davidson's formulation is identical to the conventional quasilinear theory of equations 3.12, 3.13 and 3.14 except that the definitions of the spectral energy density (equation 3.12 in the conventional theory and equation 3.31 in Davidson's) are different. In the conventional model, an initial electric field fluctuation level is specified, whereas in Davidson's model it is an initial pair correlation. It is not obvious how the two relate to one another. In particular, Davidson's spectral energy density depends explicitly on the plasma dielectric function.

3.3 Comparison of Quasilinear and Kinetic Theories

In section 2.3 it was shown that although the component instability-enhanced collision operators $C_{\rm IE}(f_s, f_{s'})$ consist of both diffusion and drag terms, that the total instability-enhanced collision operator $C_{\rm IE}(f_s) = \sum_{s'} C_{\rm IE}(f_s, f_{s'})$ has only a diffusion term. This is because the sum over species s' of all the drag components vanishes. The resultant total instability-enhanced collision operator can be written in the same form as the effective collision operator of the quasilinear theory of equations 3.13 and 3.14, which are

$$C(f_s) = \frac{\partial}{\partial \mathbf{v}} \cdot \mathcal{D}_v \cdot \frac{\partial f_s}{\partial \mathbf{v}} \tag{3.32}$$

and

$$\mathcal{D}_v \equiv \frac{q_s^2}{m_s^2} 8\pi \sum_j \int d^3k \, \frac{\mathbf{kk}}{k^2} \frac{\gamma_j \, \mathcal{E}_j(\mathbf{k})}{(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v})^2 + \gamma_j^2}. \tag{3.33}$$

However, in each of the three cases: conventional quasilinear theory, Davidson's quasilinear theory and in the instability-enhanced kinetic theory, the spectral energy density is defined differently. In conventional quasilinear theory it is

$$\mathcal{E}_j^{\rm ql}(\mathbf{k}) = \frac{|\hat{E}_1(\mathbf{k}, t=0)|^2}{(2\pi)^3 V} \frac{e^{2\gamma_j t}}{8\pi}, \tag{3.34}$$

in Davidson's quasilinear theory it is

$$\mathcal{E}_j^{\rm dv}(\mathbf{k}) = \sum_{s'} \frac{q_{s'}^2}{4\pi^2 k^2 \left|\partial\hat{\varepsilon}(\mathbf{k},\omega)/\partial\omega\right|^2_{\omega_j}} \int d^3v \int d^3v' \frac{\tilde{P}(\mathbf{k}, \mathbf{v}, \mathbf{v}', t=0) e^{2\gamma_j t}}{(\omega_j - \mathbf{k}\cdot\mathbf{v})(\omega_j^* - \mathbf{k}\cdot\mathbf{v}')}, \tag{3.35}$$

and in the instability-enhanced term of the kinetic theory, it is

$$\mathcal{E}_j^{\rm kin}(\mathbf{k}) = \sum_{s'} \frac{q_{s'}^2}{4\pi^2 k^2 \left|\partial\hat{\varepsilon}(\mathbf{k},\omega)/\partial\omega\right|^2_{\omega_j}} \int d^3v' \frac{f_{s'}(\mathbf{v}') \, e^{2\gamma_j t}}{(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v}')^2 + \gamma_j^2}. \tag{3.36}$$

An important feature of equation 3.36 is that it does not depend on specifying an initial electrostatic fluctuation level, as equation 3.34 requires, or an initial pair correlation function, as equation 3.35 requires. This is because the source of fluctuations, which the spectral energy density describes, is self-consistently accounted for from discrete particle motion in the plasma. In the quasilinear theories, the fluctuation source in not specified. Equation 3.36 also shows that when fluctuations originate from discrete particle motion, the spectral energy density has a particular dependence on $\mathbf{k}$ that is determined by the plasma dielectric function. This $\mathbf{k}$ dependence cannot be captured by the conventional quasilinear

theory, equation 3.34, which typically proceeds by specifying a constant for $|\hat{E}_1(t=0)|^2$ to determine the spectral energy density.

3.4 Physical Properties of the Collision Operator

The kinetic equation derived in chapter 2 obeys certain physical properties such as conservation laws and the Boltzmann $\mathcal{H}$-theorem. An overview of these properties is provided in this section along with a discussion of how the plasma evolves to equilibrium and how these properties relate to those of the effective collision operator in conventional quasilinear theory. The analogous properties for the stable plasma case, when $C_{\rm IE}$ is negligible compared to $C_{\rm LB}$, were first provided by Lenard [2] and many of the derivations that follow in this section are similar to what was given in his paper.

3.4.1 Density Conservation

Collisions do not create or destroy particles or cause them to change species. For collisions between species s and s', this can be expressed mathematically as

$$\int d^3v\, C(f_s, f_{s'}) = 0, \tag{3.37}$$

which also implies the less restrictive conditions that the species s density is conserved $\int d^3v df_s/dt = \int d^3v \sum_{s'} C(f_s, f_{s'}) = 0$ and the total plasma density is conserved $\int d^3v \sum_s df_s/dt = 0$. This is true of both the $C_{\rm LB}$ and $C_{\rm IE}$ terms individually, since each can be written as a velocity-space divergence.

Proof: Equation 3.37 follows directly from writing $C(f_s, f_{s'})$ in the form of a divergence of the collisional current $C(f_s, f_{s'}) = -\nabla_v \cdot \mathbf{J}_v^{s/s'}$. The integral over velocity space vanishes due to the divergence theorem since $\mathbf{J}_v^{s/s'}$ is zero at infinite velocity.

The conventional quasilinear theory, summarized in section 3.1, does not distinguish each of the species s', so one cannot show that equation 3.37 is satisfied; recall that there is only a total collision operator $C(f_s)$ in the quasilinear theory, not individual collision operators $C(f_s, f_{s'})$. In section 3.5 we discuss why it is not possible to adapt quasilinear theory to describe component collisions. However, the total effective collision operator in quasilinear theory can be written in the form of a velocity-space divergence, and thus it does satisfy that the species s and total plasma density are conserved.

3.4.2 Momentum Conservation

Momentum lost from species s due to collisions of species s with species s' is gained by species s'. Mathematically, this is expressed as

$$\int d^3v\, m_s \mathbf{v} C(f_s, f_{s'}) + \int d^3v\, m_{s'} \mathbf{v} C(f_{s'}, f_s) = 0. \tag{3.38}$$

Equation 3.38 implies that the total momentum is conserved: $\int d^3v \sum_s m_s \mathbf{v} df_s/dt = 0$.

Proof: Equation 3.38 follows from first integrating by parts to show that $\int d^3v\ m_s \mathbf{v} C(f_s, f_{s'}) = m_s \int d^3v \mathbf{J}_v^{s/s'}$, which is

$$\int d^3v\ m_s \mathbf{v} C(f_s, f_{s'}) = -\int d^3v \int d^3v' m_s \mathcal{Q}_{s,s'} \cdot \mathcal{X}_{s,s'}, \tag{3.39}$$

where we have defined

$$\mathcal{X}_{s,s'}(\mathbf{v}, \mathbf{v}') = \frac{f_{s'}(\mathbf{v}')}{m_s} \frac{\partial f_s(\mathbf{v})}{\partial \mathbf{v}} - \frac{f_s(\mathbf{v})}{m_{s'}} \frac{\partial f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'}. \tag{3.40}$$

An expression for $\int d^3v\ m_{s'} \mathbf{v} C(f_{s'}, f_s)$ is obtained by the substitutions $s \leftrightarrow s'$ and $\mathbf{v} \leftrightarrow \mathbf{v}'$ in equation 3.39. Using the properties $m_{s'} \mathcal{Q}_{s',s} = m_s \mathcal{Q}_{s,s'}$ and $\mathcal{X}_{s,s'}(\mathbf{v}, \mathbf{v}') = -\mathcal{X}_{s',s}(\mathbf{v}', \mathbf{v})$ in the result and adding it to equation 3.39 yields the conservation of momentum expression of equation 3.38.

Since conventional quasilinear theory does not resolve component collision operators, it does not satisfy equation 3.38. In quasilinear theory, only the total plasma momentum can be shown to be conserved. To show this, first note that

$$\begin{aligned} \int d^3v \sum_s m_s \mathbf{v} \frac{df_s}{dt} &= \sum_s m_s \int d^3v\, \mathcal{D}_v \cdot \frac{\partial f_s}{\partial \mathbf{v}} \\ &= -\frac{1}{2} \sum_j \int d^3k\, \mathbf{k}\, \mathcal{E}_j^{\mathrm{ql}}(\mathbf{k}) \sum_s \frac{4\pi q_s^2}{k^2 m_s} \int d^3v \frac{(-\gamma_j) \mathbf{k} \cdot \partial f_s/\partial \mathbf{v}}{(\omega_{R,j} - \mathbf{k} \cdot \mathbf{v})^2 + \gamma_j^2}, \end{aligned} \tag{3.41}$$

in which we have first integrated by parts, then substituted equation 3.16 for $\mathcal{D}_v$. From equation 2.125, the definition of ω_j implies that

$$\Im\{\hat{\varepsilon}(\mathbf{k}, \omega_j)\} = \sum_s \frac{4\pi q_s^2}{k^2 m_s} \int d^3v \frac{(-\gamma_j) \mathbf{k} \cdot \partial f_s/\partial \mathbf{v}}{(\omega_{R,j} - \mathbf{k} \cdot \mathbf{v})^2 + \gamma_j^2} = 0. \tag{3.42}$$

Thus, $\int d^3v \sum_s m_s \mathbf{v} df_s/dt \propto \Im\{\hat{\varepsilon}(\mathbf{k}, \omega_j)\} = 0$ proves that total momentum is conserved in conventional quasilinear theory.

3.4.3 Energy Conservation

The sum of particle and wave energy is conserved. The energy lost by species s due to conventional Coulomb collisions of s with s' (described by the Lenard-Balescu operator) is gained by s'. Mathematically this can be written

$$\int d^3v \frac{1}{2} m_s v^2 C_{\mathrm{LB}}(f_s, f_{s'}) + \int d^3v \frac{1}{2} m_{s'} v^2 C_{\mathrm{LB}}(f_{s'}, f_s) = 0, \tag{3.43}$$

and it implies that $\int d^3v \sum_s m_s v^2 C_{\mathrm{LB}}(f_s)/2 = 0$. The instability-enhanced portion of the collision operator shows that a change in total energy in the plasma is balanced by a change in wave energy. Thus we find that the total energy conservation relation is given by

$$\int d^3v \sum_s \frac{1}{2} m_s v^2 C(f_s) = -\frac{\partial}{\partial t} \int d^3k \frac{\mathcal{E}(\mathbf{k})}{k^2} \tag{3.44}$$

in which the spectral energy density is defined in terms of equation 3.36. Equation 3.44 is also satisfied in conventional quasilinear theory (if equation 3.34, instead of 3.36, is used to define the spectral energy density).

Proof: Conservation of energy from the Lenard-Balescu collision operator, equation 3.43, follows from first integrating by parts to show $\int d^3v\, m_s v^2 C(f_s, f_{s'})/2 = \int d^3v\, m_s \mathbf{v} \cdot \mathbf{J}_v^{s/s'}$. Putting in $\mathbf{J}_{\mathrm{LB}}^{s/s'}$ and using the same method that was used in the proof of momentum conservation for obtaining an expression for $\int d^3v\, m_{s'} v^2 C_{\mathrm{LB}}(f_{s'}, f_s)/2$ yields

$$\int d^3v \frac{v^2}{2} \left[m_s C_{\mathrm{LB}}(f_s, f_{s'}) + m_{s'} C_{\mathrm{LB}}(f_{s'}, f_s) \right] = -\int d^3v \int d^3v' m_s \mathcal{Q}_{\mathrm{LB}} \cdot (\mathbf{v} - \mathbf{v}') \cdot \mathcal{X}_{s,s'}. \tag{3.45}$$

Since $\mathcal{Q}_{\mathrm{LB}} \cdot (\mathbf{v} - \mathbf{v}') = 0$, the right side of equation 3.45 vanishes, which proves equation 3.43 and by a trivial extension $\int d^3v \sum_s m_s v^2 C_{\mathrm{LB}}(f_s)/2 = 0$.

The only nonvanishing component of the conservation of energy relation is the instability-enhanced portion which can be written $\int d^3v \sum_s m_s v^2 C_{\mathrm{IE}}(f_s)/2 = -\int d^3v \sum_s m_s \mathbf{v} \cdot \mathcal{D}_{\mathrm{IE,diff}} \cdot \partial f_s/\partial \mathbf{v}$. Inserting $\mathcal{D}_{\mathrm{IE,diff}}$ from equation 2.122, identifying equations 3.33 and 3.36, gives

$$\int d^3v \sum_s \frac{1}{2} m_s v^2 C_{\mathrm{IE}}(f_s) = -\int d^3k \frac{2\gamma \mathcal{E}(\mathbf{k})}{k^2} \left[\sum_s \frac{4\pi q_s^2}{k^2 m_s} \int d^3v \frac{\mathbf{k} \cdot \mathbf{v}\, \mathbf{k} \cdot \partial f_s/\partial \mathbf{v}}{(\omega_{R,j}^2 - \mathbf{k} \cdot \mathbf{v})^2 + \gamma_j^2} \right]. \tag{3.46}$$

Equation 2.126, describing $\Re\{\hat{\varepsilon}(\mathbf{k}, \omega_j)\} = 0$, shows that the term in square brackets in equation 3.46 is equal to 1. Identifying $2\gamma_j \mathcal{E} = \partial \mathcal{E}/\partial t$ in equation 3.46, from equation 3.36, and adding the condition $\int d^3v \sum_s m_s v^2 C_{\mathrm{LB}}(f_s)/2 = 0$ to the result completes the proof of equation 3.44.

Finally, it is interesting to point out that if the approximate expression 2.47 for $\mathcal{Q}_{\rm IE}$ is used in place of the more fundamental expression 2.44, then $\mathcal{Q}_{\rm IE}\cdot(\mathbf{v}-\mathbf{v}')=0$ implies that conservation of energy for collisions between individual species, i.e. equation 3.43, can also be extended to the instability enhanced term. In this case, the change in wave energy is missed. Making the approximation of equation 2.47 leads to neglect of the change in wave energy, which is a result of the wave energy being much smaller than the plasma energy. This demonstrates an instance where caution must be taken when applying the δ-function approximation to equation 2.44, and using the result, which is equation 2.47.

3.4.4 Positive-Definiteness of f_s

If $f_s(\mathbf{v})\geq 0$ initially, $f_s(\mathbf{v})\geq 0$ for all time. This property is meant to check that unphysical distribution functions (that have a negative number of particles for some velocity) cannot arise as an initially physically meaningful $f_s(\mathbf{v})$ evolves in time.

Proof: The proof of this for $C(f_s)=C_{\rm LB}(f_s)+C_{\rm IE}(f_s)$ is essentially the same as that for just $C(f_s)=C_{\rm LB}(f_s)$ that was first provided by Lenard [2]. It goes as follows: If $f_s(\mathbf{v})>0$ for all $\mathbf{v}$ initially, but at a later time $f_s(\mathbf{v})<0$ for some $\mathbf{v}$, there must be an instant when the minimum value of f_s first becomes negative. At this point in velocity-space, the following four conditions must be satisfied:

(i) $f_s(\mathbf{v})=0$ (there must be a zero of f_s itself)

(ii) $\partial f_s(\mathbf{v})/\partial\mathbf{v}=0$ (the derivative of f_s is zero at its minimum)

(iii) $\partial^2 f_s(\mathbf{v})/\partial\mathbf{v}\partial\mathbf{v}$ is a non-negative definite tensor (so that f_s will have a minimum instead of a maximum or a saddle point), and

(iv) $df_s/dt<0$ (f_s must decrease as t increases in order to become negative).

The plasma kinetic equation is

$$\begin{aligned}\frac{df_s}{dt} &= -\sum_{s'}\frac{\partial}{\partial\mathbf{v}}\cdot\int d^3v'\,\mathcal{Q}\cdot\left(\frac{f_s(\mathbf{v})}{m_{s'}}\frac{\partial f_{s'}(\mathbf{v}')}{\partial\mathbf{v}'}-\frac{f_{s'}(\mathbf{v}')}{m_s}\frac{\partial f_s(\mathbf{v})}{\partial\mathbf{v}}\right)\\ &= -\sum_{s'}\int d^3v'\left(\frac{\partial}{\partial\mathbf{v}}\cdot\mathcal{Q}\right)\cdot\left(\frac{f_s(\mathbf{v})}{m_{s'}}\frac{\partial f_{s'}(\mathbf{v}')}{\partial\mathbf{v}'}-\frac{f_{s'}(\mathbf{v}')}{m_s}\frac{\partial f_s(\mathbf{v})}{\partial\mathbf{v}}\right)\\ &\quad -\sum_{s'}\int d^3v'\,\mathcal{Q}\colon\left(\frac{1}{m_{s'}}\frac{\partial f_s(\mathbf{v})}{\partial\mathbf{v}}\frac{\partial f_{s'}(\mathbf{v}')}{\partial\mathbf{v}'}-\frac{f_{s'}(\mathbf{v}')}{m_s}\frac{\partial^2 f_s(\mathbf{v})}{\partial\mathbf{v}\partial\mathbf{v}}\right).\end{aligned}\tag{3.47}$$

Applying (i) and (ii) at the point of interest in velocity space, this reduces to

$$\frac{df_s}{dt} = \sum_{s'} \int d^3v' \frac{f_{s'}(\mathbf{v}')}{m_s} \mathcal{Q} \colon \frac{\partial^2 f_s(\mathbf{v})}{\partial \mathbf{v} \partial \mathbf{v}}. \tag{3.48}$$

However, $\mathcal{Q} = \mathcal{Q}_{\rm LB} + \mathcal{Q}_{\rm IE}$ is a non-negative definite tensor and, from (iii), so is $\partial^2 f_s / \partial \mathbf{v} \partial \mathbf{v}$. Thus, equation 3.48 implies that $df/dt \geq 0$, which contradicts condition (iv). Hence, if $f_s \geq 0$ initially, then $f_s \geq 0$ for all time.

For conventional quasilinear theory, since $\mathcal{D}_v$ (equation 3.16) is also a non-negative definite tensor when the plasma is unstable ($\gamma_j \geq 0$), the positive definiteness of f_s is also obeyed in quasilinear theory. However, one criticism of quasilinear theory is that it does not transition to stable plasmas [69]. If $\gamma_j < 0$, the diffusion coefficient can have negative components and an unphysical equation results, with the consequence that f_s can become negative. This would be a misapplication of quasilinear theory because it is based on the assumption that the plasma is unstable; but it also illustrates the limits of quasilinear theory because a Coulomb collision operator [i.e., $C_{\rm LB}(f_s)$] is required near marginal stability. In the kinetic theory, when γ is near zero (or negative), the Lenard-Balescu term dominates, which ensures that the total $\mathcal{Q}$ remains non-negative-definite regardless of whether instabilities are present.

3.4.5 Galilean Invariance

The component collision operators $C(f_s, f_{s'})$, and consequently the total collision operator $C(f_s)$ are Galilean invariant.

Proof: Transforming coordinates to $\mathbf{w} \equiv \mathbf{v} - \mathbf{V_f}$ and $\mathbf{w}' \equiv \mathbf{v}' - \mathbf{V_f}$ introduces a Doppler shift into equations 2.16, 2.17 and 2.18 where $\omega \leftrightarrow \omega + \mathbf{k} \cdot \mathbf{V_f}$ when $\mathbf{v} \leftrightarrow \mathbf{w}$. By defining the variables

$$\bar{\omega} \equiv \omega + \mathbf{k} \cdot \mathbf{V_f} \quad \text{and} \quad \bar{\omega}' \equiv \omega' + \mathbf{k} \cdot \mathbf{V_f} \tag{3.49}$$

which satisfy $d^3\omega = d^3\bar{\omega}$ and $\partial/\partial\bar{\omega} = \partial/\partial\omega$, we can replace ω with $\bar{\omega}$, ω' with $\bar{\omega}'$, $\mathbf{v}$ with $\mathbf{w}$, $\mathbf{v}'$ with $\mathbf{w}'$ and the entire analysis of section 2.1 can be repeated in these new coordinates. Thus, the collision operator, equation 2.45, is Galilean invariant. Therefore the kernel satisfies Galilean invariance, $Q(\mathbf{v}, \mathbf{v}') = Q(\mathbf{w}, \mathbf{w}')$, as well.

3.4.6 The Boltzmann $\mathcal{H}$-Theorem

The $\mathcal{H}$-functional for each species s is defined as $\mathcal{H}_s \equiv \int d^3v\, f_s(\mathbf{v}) \ln f_s(\mathbf{v})$ and the total $\mathcal{H}$ is the sum of the component species $\mathcal{H} = \sum_s \mathcal{H}_s$. The Boltzmann $\mathcal{H}$-theorem states that the total $\mathcal{H}$ satisfies $d\mathcal{H}/dt \leq 0$. It is equivalent to stating that entropy always increases until equilibrium in reached.

Proof: The time derivative of $\mathcal{H}_s$ is $d\mathcal{H}_s/dt = \int d^3v \big[1 + \ln f_s(\mathbf{v})\big] df_s(\mathbf{v})/dt$. Using the conservation of density property from section 3.4.1 gives $d\mathcal{H}_s/dt = -\int d^3v \sum_{s'} \ln(f_s)(\partial/\partial\mathbf{v}) \cdot \mathbf{J}_v^{s/s'}$. Integrating by parts yields $d\mathcal{H}_s/dt = \sum_{s'} \int d^3v\, \mathbf{J}_v^{s/s'} \cdot \partial \ln f_s/\partial\mathbf{v}$. We then identify the components of $\mathcal{H}_s$ such that $\mathcal{H}_s = \sum_{s'} \mathcal{H}_{s,s'}$. Putting in the kinetic equation 2.45 gives

$$\frac{d\mathcal{H}_{s,s'}}{dt} = -\int d^3v \int d^3v' \frac{1}{m_s} \frac{\partial \ln f_s(\mathbf{v})}{\partial \mathbf{v}} \cdot \big(m_s \mathcal{Q}_{s,s'}\big) \cdot \mathcal{X}_{s,s'}. \tag{3.50}$$

By interchanging the species $s \leftrightarrow s'$ and dummy integration variables $\mathbf{v} \leftrightarrow \mathbf{v}'$ an expression for $\mathcal{H}_{s',s}$ is obtained

$$\frac{d\mathcal{H}_{s',s}}{dt} = \int d^3v \int d^3v' \frac{1}{m_{s'}} \frac{\partial \ln f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'} \cdot \big(m_{s'} \mathcal{Q}_{s',s}\big) \cdot \mathcal{X}_{s,s'}, \tag{3.51}$$

in which we have used $\mathcal{X}_{s',s} = -\mathcal{X}_{s,s'}$. Using $m_s \mathcal{Q}_{s,s'} = m_{s'} \mathcal{Q}_{s',s}$ in equation 3.51, along with equation 3.50, in $2\mathcal{H} = \sum_s \sum_{s'} \big(\mathcal{H}_{s,s'} + \mathcal{H}_{s',s}\big)$ yields

$$2\frac{d\mathcal{H}}{dt} = -\sum_s \sum_{s'} \int d^3v \int d^3v' \frac{\mathcal{X}_{s,s'} \cdot \big(m_s \mathcal{Q}_{s,s'}\big) \cdot \mathcal{X}_{s,s'}}{f_s(\mathbf{v}) f_{s'}(\mathbf{v}')}. \tag{3.52}$$

Since the $\mathcal{Q}_{s,s'} = \mathcal{Q}_{\mathrm{LB}} + \mathcal{Q}_{\mathrm{IE}}$ (from equations 2.43 and 2.44) is positive-semidefinite (a non-negative-definite tensor) and $f_s, f_{s'} \geq 0$ (see section 3.4.4), each term on the right side of equation 3.52 is negative-semidefinite. Thus, we find that the Boltzmann $\mathcal{H}$-theorem is satisfied: $d\mathcal{H}/dt \leq 0$.

3.4.7 Uniqueness of Maxwellian Equilibrium

The unique equilibrium distribution function is Maxwellian and the approach to equilibrium is hastened by instabilities. Equilibrium is established when $d\mathcal{H}/dt = 0$. In the analysis below, we first show from C_{LB} that the unique equilibrium state of a plasma is a Maxwellian in which each species has the same temperature and flow velocity. Since instabilities require a free energy source, they cannot be present near thermodynamic equilibrium. Thus, the final equilibrium state of the plasma is determined by C_{LB}. However, bounded plasmas are rarely in true thermodynamic equilibrium. A much more common

concern is to determine the timescales for which equilibration between individual species occurs. The fastest timescales are typically for self-equilibration within species. For example, electrons and ions in a plasma may be in equilibrium with themselves, in which case $d\mathcal{H}_{e,e}/dt = 0$ and $d\mathcal{H}_{i,i}/dt = 0$, but not in equilibrium with each other, so $d\mathcal{H}_{i,e}/dt \neq 0$. In this case, electrons and ions will individually have Maxwellian distributions, but their temperature and flow velocities will not necessarily be the same and instabilities may be present. In the second part of the analysis below, we show that instabilities can significantly shorten the timescale for which individual species reach a unique Maxwellian equilibrium (such that $d\mathcal{H}_{s,s}/dt = 0$, but where $d\mathcal{H}_{s,s'}/dt$ is not necessarily 0 for all s').

Analysis: First, we consider the final equilibrium state of the plasma from the C_{LB} term of the collision operator. Since each term of equation 3.52 is negative-semidefinite, each must vanish independently in order to reach equilibrium at $d\mathcal{H}/dt = 0$. The terms that tend to zero on the fastest time scale are those describing like-particle collisions $s = s'$. Considering these first, $d\mathcal{H}_{s,s}/dt = 0$ implies that $\mathcal{X}_{s,s} \propto \mathbf{v} - \mathbf{v}'$ [because $\mathcal{Q}_{\mathrm{LB}} \cdot (\mathbf{v} - \mathbf{v}') = 0$]. From equation 3.40, we have

$$\mathcal{X}_{s,s} = \frac{f_s(\mathbf{v}) f_s(\mathbf{v}')}{m_s} \left[\frac{\partial \ln f_s(\mathbf{v})}{\partial \mathbf{v}} - \frac{\partial \ln f_s(\mathbf{v}')}{\partial \mathbf{v}'} \right]. \tag{3.53}$$

Thus, the condition $\mathcal{X}_{s,s} \propto \mathbf{v} - \mathbf{v}'$ implies

$$\frac{\partial \ln f_s(\mathbf{v})}{\partial \mathbf{v}} - \frac{\partial \ln f_s(\mathbf{v}')}{\partial \mathbf{v}'} = -A(\mathbf{v}, \mathbf{v}')(\mathbf{v} - \mathbf{v}') \tag{3.54}$$

in which A is some function of $\mathbf{v}$ and $\mathbf{v}'$. Applying the curls $(\partial/\partial\mathbf{v})\times$ and, separately $(\partial/\partial\mathbf{v}')\times$, to equation 3.54 yields

$$\frac{\partial A}{\partial \mathbf{v}} \times (\mathbf{v} - \mathbf{v}') = 0 \quad \text{and} \quad \frac{\partial A}{\partial \mathbf{v}'} \times (\mathbf{v} - \mathbf{v}') = 0. \tag{3.55}$$

Thus, given $\mathbf{v}$ or $\mathbf{v}'$, A is spherically symmetric in $\mathbf{v} - \mathbf{v}'$. Furthermore, equation 3.54 shows that $A(\mathbf{v}, \mathbf{v}') = A(\mathbf{v}', \mathbf{v})$. This, combined with the spherically symmetric property, implies that A depends only on the distance between the two points in velocity space

$$A = A(|\mathbf{v} - \mathbf{v}'|). \tag{3.56}$$

By setting $\mathbf{v} = 0$, then $\mathbf{v}' = 0$, equations 3.54 and 3.56 imply

$$\frac{\partial \ln f_s(\mathbf{v}')}{\partial \mathbf{v}'} = -\mathbf{v}' A(|\mathbf{v}'|) + \mathbf{K}_1 \quad \text{and} \quad \frac{\partial \ln f_s(\mathbf{v})}{\partial \mathbf{v}} = -\mathbf{v} A(|\mathbf{v}|) + \mathbf{K}_2, \tag{3.57}$$

in which $\mathbf{K}_1 = \partial \ln f_s(\mathbf{v}')/\partial \mathbf{v}'|_{\mathbf{v}'=0}$ and $\mathbf{K}_2 = \partial \ln f_s(\mathbf{v})/\partial \mathbf{v}|_{\mathbf{v}=0}$ are constants. Putting equation 3.57 into equation 3.54 gives

$$\mathbf{v}A(|\mathbf{v}|) - \mathbf{v}'A(|\mathbf{v}'|) = (\mathbf{v} - \mathbf{v}')A(|\mathbf{v} - \mathbf{v}'|) + (\mathbf{K}_2 - \mathbf{K}_1). \tag{3.58}$$

Considering the velocity-dependent parts, the only continuous solution of this is A = constant and $\mathbf{K}_1 = \mathbf{K}_2$. We will define $\mathbf{K}_1 = \mathbf{K}_2 = \mathbf{B}$. Then, equation 3.57 gives

$$\frac{\partial \ln f_s(\mathbf{v})}{\partial \mathbf{v}} + A\,\mathbf{v} = \mathbf{B}. \tag{3.59}$$

Integrating equation 3.59 shows that $f_s(\mathbf{v})$ has the general Maxwellian form $f_{Ms}(\mathbf{v}) = \exp\big(-Av^2/2 + \mathbf{B}\cdot\mathbf{v} + C\big)$, where C is an additional constant of integration.

Applying the conventional definitions for density $n_s = \int d^3v f_s$, flow velocity $\mathbf{V}_s = \int d^3v \mathbf{v} f_s/n_s$ and thermal speed $v_{Ts}^2 = \frac{2}{3}\int d^3v(\mathbf{v} - \mathbf{V}_s)^2 f_s/n_s = 2T_s/m_s$, the five constants A, $\mathbf{B}$ and C can be written in terms of n_s, $\mathbf{V}_s$ and T_s. These definitions yield: $A = m_s/(2T_s)$, $\mathbf{B} = m_s\mathbf{V}_s/T_s$ and $\exp(-C) = n_s/(\pi^{3/2}v_{Ts}^3)\exp(-V_s^2/v_{Ts}^2)$. Then, the Maxwellian for species s can be written in the familiar form

$$\boxed{f_{Ms}(\mathbf{v}) = \frac{n_s}{\pi^{3/2}v_{Ts}^3}\exp\left[-\frac{(\mathbf{v} - \mathbf{V}_s)^2}{v_{Ts}^2}\right].} \tag{3.60}$$

On a longer time scale the unlike particle terms ($s \neq s'$) of equation 3.52 must also vanish for equilibrium to be reached. This implies $\mathcal{X}_{s,s'} \propto \mathbf{v} - \mathbf{v}'$. Putting the individual species Maxwellians of equation 3.60 into this condition gives

$$\left(\frac{\mathbf{v}}{T_s} - \frac{\mathbf{v}'}{T_{s'}}\right) + \left(\frac{\mathbf{V}_{s'}}{T_{s'}} - \frac{\mathbf{V}_s}{T_s}\right) \propto \mathbf{v} - \mathbf{v}', \tag{3.61}$$

which is satisfied only if $T_s = T_{s'}$ and $\mathbf{V}_s = \mathbf{V}_{s'}$. Thus, the unique equilibrium state of the plasma is that each species have a Maxwellian distribution of the form of equation 3.60 with the same flow velocity and temperature.

Next, we consider the role of instability-enhanced collisions in the equilibration process. The analysis just considered could be repeated by substituting $\mathcal{Q}_{\mathrm{IE}}$ for $\mathcal{Q}_{\mathrm{LB}}$, except that unlike $\mathcal{Q}_{\mathrm{LB}}$, $\mathcal{Q}_{\mathrm{IE}}$ (equation 2.44) is not proportional to $\delta[\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}')]$ and does not satisfy $\mathcal{Q}_{\mathrm{IE}}\cdot(\mathbf{v} - \mathbf{v}') = 0$. However, $\mathcal{Q}_{\mathrm{IE}}$ has a Lorentzian form in velocity space that is very peaked around $\mathbf{k}\cdot(\mathbf{v} - \mathbf{v}') = 0$ and the dominant term can be written in the delta function form. Substituting $\mathcal{Q}_{\mathrm{IE}}$ into equation 3.52 gives an expression

that depends on velocity-space integrals in $\mathbf{v}$ and $\mathbf{v}'$ over Lorentzian distributions. We assume that the instabilities are weakly growing and thus satisfy $\gamma_j \ll \omega_{R,j} - \mathbf{k}\cdot\mathbf{v}$. These velocity-space integrals are of the form $\int dx\, g(x)\Delta/[(x-a)^2+\Delta^2]$ where $\Delta \ll a$. Here the appropriate substitutions are $\Delta = \gamma_j$, $a = \omega_{R,j}$ and $x = \mathbf{k}\cdot\mathbf{v}$ (or $x = \mathbf{k}\cdot\mathbf{v}'$). They can be approximated by

$$\int_{-\infty}^{\infty} dx\, g(x)\frac{\Delta}{(a-x)^2+\Delta^2} \approx \int_{-n\Delta}^{n\Delta} dy\, g(y+a)\frac{\Delta}{y^2+\Delta^2} \tag{3.62}$$

where n is a number large enough to span most of the integrand. Expanding $g(x)$ about the peak at $x = a$, $g(x) \approx g(a) + g'(a)y + g''(a)y^2/2 + \ldots$, the lowest order term in equation 3.62 gives $g(a)2\arctan(n) \approx \pi g(a)$. The second term is zero. The third term gives $g''(a)\Delta^2[n - \arctan(n)] \sim \mathcal{O}(\Delta^2/a^2)$. Thus, these integrals satisfy

$$\int dx\, g(x)\frac{\Delta}{(a-x)^2+\Delta^2} \approx \int dx\, \pi g(x)\delta(x-a) + \mathcal{O}\left(\frac{\Delta^2}{a^2}\right). \tag{3.63}$$

Using equation 3.63 and the property $\delta(x-a)\delta(x-b) = \delta(x-a)\delta(a-b)$, we find that within the velocity-space integrals of equation 3.52, the $\mathcal{Q}_{\mathrm{IE}}$ term can be written in the form

$$\mathcal{Q}_{\mathrm{IE}} = \sum_j \frac{2q_s^2 q_{s'}^2}{m_s}\int d^3k \frac{\mathbf{kk}}{k^4}\frac{\pi\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]\delta(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v})e^{2\gamma_j t}}{\gamma_j|\partial\hat{\varepsilon}(\mathbf{k},\omega)/\partial\omega|^2_{\omega_j}} + \mathcal{O}\left(\frac{\gamma_j^2}{\omega_{R,j}^2}\right). \tag{3.64}$$

Since the lowest order term of equation 3.64 satisfies $\mathcal{Q}_{\mathrm{IE}}\cdot(\mathbf{v}-\mathbf{v}') = 0$, one can repeat the analysis used for the Lenard-Balescu term to show that the instability-enhanced term drives the distribution of individual species to unique Maxwellian distributions (from like particle, $s = s$ collisions). The correction terms in equation 3.64 will not obey this property and can be expected to cause some deviation from a Maxwellian. However, if the distribution is non-Maxwellian, the lowest order term in equation 3.64 drives the distribution toward a Maxwellian on a faster timescale, by a factor of $\mathcal{O}(\omega_{R,j}^2/\gamma_j^2)$, than the correction terms cause a deviation from Maxwellian. Thus, if instabilities are present from the interaction of two different species, for example flow-driven instabilities, the instabilities can shorten the timescale for which each species will self-equilibrate to a Maxwellian.

On a longer time scale, the different species will also equilibrate with one another. For the electron-ion plasma example, and if $T_i \approx T_e$, the fastest equilibration timescale is the electron-electron. The ion-ion collision frequency is a bit smaller at $\nu^{i-i} \sim \nu^{e-e}\sqrt{m_e/M_i}$, and at the slowest timescale, the ions and electrons equilibrate with one another according to the frequency $\nu^{i-e} \sim \nu^{e-e}(m_e/M_i)$. These scalings are obtained directly from $\nu^{s-s'} \sim C(f_s, f_{s'})/f_s$ and are the same for both the Lenard-Balescu

term and the instability-enhanced term. Before the final equilibrium state can be established (in which each species is Maxwellian with the same temperature and flow velocity) the plasma can be in a state where individual species have come to be in Maxwellian equilibrium, but where the flow speed and temperature of different species have not yet equilibrated. In this state, instabilities can persist. In chapters 4, 5 and 6, applications are considered where flow-driven instabilities cause like-particle species to become Maxwellian, but where the timescale for different species to equilibrate with one another (where the flow velocities and temperatures would be the same) remain much longer than the time it takes the plasma to flow out of the system.

The fact that $\mathcal{Q}_{\rm IE} \cdot (\mathbf{v} - \mathbf{v}')$ is only approximately zero is not unique to the instability-enhanced term. Recall from section 1.1.1 that $\mathcal{Q}_{\rm LB}$ has components that are $\mathcal{O}(1/\ln\Lambda)$ which are neglected. These also apparently lead to modifications to the equilibrium because the $\hat{z}\hat{z}$ component of equation 1.60 will prevent the strict $\mathcal{Q}_{\rm LB} \cdot (\mathbf{v} - \mathbf{v}') = 0$ property. The $\mathcal{O}(\gamma^2/\omega_{R,j}^2)$ terms are typically even smaller than $1/\ln\Lambda$. Thus, the property of a unique Maxwellian equilibrium is obeyed as strictly by the instability-enhanced term, in a weakly unstable plasma, as it is by the Lenard-Balescu term.

It may also be worth pointing out that one cannot show from conventional quasilinear theory that instability-enhanced collisions drive the plasma to a unique Maxwellian equilibrium. This can be shown within the kinetic theory because it distinguishes the origin of fluctuations and thus determines the spectral energy density $\mathcal{E}(\mathbf{k})$ (see equation 3.36). Specification of $\mathcal{E}(\mathbf{k})$ is required in order to show that $\mathcal{Q}_{\rm IE}$ can be written in a form proportional to $\delta[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')]$. This property is required in order to show that Maxwellian is the unique equilibrium.

3.5 Attempt to Reconcile Elements of Kinetic and Quasilinear Theories

Throughout this chapter, we have discussed similarities and differences between the kinetic theory of chapter 2 and conventional quasilinear theory (section 3.1). Although the basic equations of each theory look formally similar, we have seen that the resultant evolution equations have important differences that also effect the physical properties that each equation obeys. One difference is that the kinetic

approach associates the source of fluctuations with discrete particle motion, while quasilinear theory does not specify the source of fluctuations. A related difference is that the ensemble average in kinetic theory is an average over initial particle positions in the six-dimensional $(\mathbf{x}, \mathbf{v})$ phase-space, while the quasilinear average is simply a spatial average. The kinetic theory uses Gauss's law and the dielectric function $\hat{\varepsilon}$ consequently appears self-consistently in the analysis. In quasilinear theory, Gauss's law is not explicitly used. It must, however, be used externally in order to determine the dispersion relations, which determine the ω_j that are assumed from the outset of the model derivation.

A result of the different approaches is that the kinetic theory distinguishes individual collision operators that describe the collisional interaction between individual species s and s'. The quasilinear approach can only access a total collision operator because the fluctuation source is unknown. We have seen in section 3.4 that the ability of the kinetic theory to distinguish individual collision operators leads to more restrictive conservation laws, and is a necessary component in proving that a Maxwellian is the unique equilibrium when instabilities are present. In this section, we address the question: is there a way to reformulate conventional quasilinear theory so that individual species interactions are resolved, without necessarily restricting the discussion to instabilities that originate from a discrete particle source (which the kinetic theory assumes)? We will find that the answer is no. Although this is not possible, the question is often raised when this work is discussed, and it is instructive to show why conventional quasilinear is unable to distinguish individual species collision operators.

A major difference in the mathematics used in the kinetic and quasilinear derivations is that the kinetic approach uses formal definitions for Fourier and Laplace integral transforms, while the quasilinear theory assumes that first-order quantities in the perturbation scheme satisfy $f_1 = \sum_j \bar{f}_1 \exp(-i\omega_j t)$ (see equation 3.8) in which the ω_j are roots of the dielectric function $\hat{\varepsilon}(\mathbf{k}, \omega_j) = 0$. The dielectric function does not arise internal to the theory, but is assumed to be determined from a proper Fourier-Laplace transform solution of the Vlasov equation. Instead of making this assumption, we will use the same Fourier and Laplace integral transforms, defined in equation 2.14, that were used in the kinetic approach of chapter 2 to solve the quasilinear equations.

Recall that the quasilinear equations consist of the evolution equation

$$\frac{\partial f_{so}}{\partial \bar{t}} + \mathbf{v} \cdot \frac{\partial f_{so}}{\partial \bar{\mathbf{x}}} = -\frac{\partial}{\partial \mathbf{v}} \cdot \frac{q_s}{m_s} \langle \mathbf{E}_1 \, f_{s1} \rangle \tag{3.65}$$

and the first-order perturbation equation

$$\frac{\partial f_{s1}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{s1}}{\partial \mathbf{x}} = -\frac{q_s}{m_s} \mathbf{E}_1 \cdot \frac{\partial f_{so}}{\partial \mathbf{v}} \tag{3.66}$$

which is used to solve for right side of equation 3.65. Unlike conventional quasilinear theory, we will also need Gauss's law

$$\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{E}_1 = 4\pi \sum_s q_s \int d^3v \, f_{s1} \tag{3.67}$$

to close the system of equations when using the integral transform technique.

Using the Fourier-Laplace integral transformation, defined in equations 2.14 and 2.15, equation 3.66 can be written

$$\hat{f}_{s1}(\mathbf{k}, \mathbf{v}, \omega) = \frac{i \, \tilde{f}_{s1}(\mathbf{k}, \mathbf{v}, t = 0)}{\omega - \mathbf{k} \cdot \mathbf{v}} - \frac{q_s}{m_s} \hat{\phi}_1 \frac{\mathbf{k} \cdot \partial f_{so} / \partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}}, \tag{3.68}$$

and equation 3.67 as

$$k^2 \, \hat{\phi}_1(\mathbf{k}, \omega) = 4\pi \sum_s q_s \int d^3v \, \hat{f}_{s1}(\mathbf{k}, \mathbf{v}, \omega), \tag{3.69}$$

in which we have applied the electrostatic approximation $\hat{\mathbf{E}}_1 = -i\mathbf{k}\hat{\phi}_1$. Putting equation 3.69 into equation 3.68 yields

$$\hat{\phi}_1(\mathbf{k}, \omega) = \sum_s \frac{4\pi q_s}{k^2 \hat{\varepsilon}(\mathbf{k}, \omega)} \int d^3v \, \frac{i \tilde{f}_{s1}(t = 0)}{\omega - \mathbf{k} \cdot \mathbf{v}}, \tag{3.70}$$

in which

$$\hat{\varepsilon}(\mathbf{k}, \omega) = 1 + \sum_s \frac{4\pi q_s^2}{k^2 m_s} \int d^3v \frac{\mathbf{k} \cdot \partial f_{so} / \partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} \tag{3.71}$$

is the dielectric function, which comes about self-consistently when using the integral transform method. Equations 3.68 and 3.70 thus yield

$$\hat{\mathbf{E}}_1(\mathbf{k}', \omega') \hat{f}_{s1}(\mathbf{k}, \omega) = \frac{i \tilde{f}_{s1}(\mathbf{k}, \mathbf{v}, t = 0)}{\omega - \mathbf{k} \cdot \mathbf{v}} \sum_{s''} \frac{4\pi q_{s''} \mathbf{k}'}{k'^2 \hat{\varepsilon}(\mathbf{k}', \omega')} \int d^3v'' \frac{\tilde{f}_{s''1}(\mathbf{k}', \mathbf{v}'', t = 0)}{\omega' - \mathbf{k}' \cdot \mathbf{v}''} \tag{3.72}$$

$$- \frac{q_s}{m_s} \frac{\mathbf{k} \cdot \partial f_{so} / \partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} \left(\sum_{s'} \frac{4\pi q_{s'}}{k^2 \hat{\varepsilon}(\mathbf{k}, \omega)} \int d^3v' \frac{\tilde{f}_{s'1}(\mathbf{k}, \mathbf{v}', t = 0)}{\omega - \mathbf{k} \cdot \mathbf{v}'} \right) \left(\sum_{s''} \frac{4\pi q_{s''}}{k'^2 \hat{\varepsilon}(\mathbf{k}', \omega')} \int d^3v'' \frac{\tilde{f}_{s''1}(\mathbf{k}', \mathbf{v}'', t = 0)}{\omega' - \mathbf{k}' \cdot \mathbf{v}''} \right).$$

In conventional quasilinear theory, the first term on the right side of equations 3.68 and 3.72 is absent; see equation 3.9. This is an artifact of using the series assumption of equation 3.8, rather than a proper integral transform. The series assumption does not account for the initial condition $f_{s1}(t = 0)$, but the integral transform does. Thus, it seems the absence of this initial condition term is a fault of the conventional quasilinear theory. One argument to justify its absence may be that it has only one

$\hat{\varepsilon}$ in its denominator, while the second term has two. This will lead to $\exp(2\gamma t)$ growth of the second term, rather than $\exp(\gamma t)$ for the first; thus the second term should be larger. However, this is not always a valid argument. In the kinetic approach of chapter 2, the first term was multiplied by one in the form of $\hat{\varepsilon}/\hat{\varepsilon}$ where the numerator is written explicitly in terms of equation 3.71. Doing the same here yields the following expression with three terms:

$$\hat{\mathbf{E}}_1(\mathbf{k}',\omega')\hat{f}_{s1}(\mathbf{k},\omega) = \tag{3.73}$$
$$\frac{i\tilde{f}_{s1}(\mathbf{k},\mathbf{v},t=0)}{\omega-\mathbf{k}\cdot\mathbf{v}}\left(\sum_{s'}\frac{4\pi q_{s'}^2}{k^2 m_{s'}\hat{\varepsilon}(\mathbf{k},\omega)}\int d^3v'\frac{\mathbf{k}\cdot\partial f_{s'o}/\partial\mathbf{v}'}{\omega-\mathbf{k}\cdot\mathbf{v}'}\right)\left(\sum_{s''}\frac{4\pi q_{s''}\mathbf{k}'}{k'^2\hat{\varepsilon}(\mathbf{k}',\omega')}\int d^3v''\frac{\tilde{f}_{s''1}(\mathbf{k}',\mathbf{v}'',t=0)}{\omega'-\mathbf{k}'\cdot\mathbf{v}''}\right)$$
$$-\frac{q_s}{m_s}\frac{\mathbf{k}\cdot\partial f_{so}/\partial\mathbf{v}}{\omega-\mathbf{k}\cdot\mathbf{v}}\left(\sum_{s'}\frac{4\pi q_{s'}}{k^2\hat{\varepsilon}(\mathbf{k},\omega)}\int d^3v'\frac{\tilde{f}_{s'1}(\mathbf{k},\mathbf{v}',t=0)}{\omega-\mathbf{k}\cdot\mathbf{v}'}\right)\left(\sum_{s''}\frac{4\pi q_{s''}}{k'^2\hat{\varepsilon}(\mathbf{k}',\omega')}\int d^3v''\frac{\tilde{f}_{s''1}(\mathbf{k}',\mathbf{v}'',t=0)}{\omega'-\mathbf{k}'\cdot\mathbf{v}''}\right)$$
$$+\frac{i\tilde{f}_{s1}(\mathbf{k},\mathbf{v},t=0)}{\omega-\mathbf{k}\cdot\mathbf{v}}\frac{1}{\hat{\varepsilon}(\mathbf{k},\omega)}\sum_{s''}\frac{4\pi q_{s''}\mathbf{k}'}{k'^2\hat{\varepsilon}(\mathbf{k}',\omega')}\int d^3v''\frac{\tilde{f}_{s''1}(\mathbf{k}',\mathbf{v}'',t=0)}{\omega'-\mathbf{k}'\cdot\mathbf{v}''}.$$

From inspection of equation 3.73, we can see explicitly how differences in the quasilinear versus kinetic formulations lead to different collision operators. Recall that there are three distinctions between the kinetic and quasilinear approaches:

(1) The kinetic theory applied Fourier-Laplace integral transforms, while conventional quasilinear theory assumes that $f_{s1}(\mathbf{x},\mathbf{v},t)=\sum_j \bar{f}_{s1}(\mathbf{x},\mathbf{v})\exp(-i\omega_j t)$ where the ω_j solve $\hat{\varepsilon}(\mathbf{k},\omega_j)=0$.

(2) The kinetic theory associates fluctuations with discrete particle motion,

$$\tilde{f}_{s1}(\mathbf{k},\mathbf{v},t=0)=\sum_{i=1}^{N}e^{-i\mathbf{k}\cdot\mathbf{x}_{io}}\delta(\mathbf{v}-\mathbf{v}_{io})-(2\pi)^3\delta(\mathbf{k})\,f_{so}(\mathbf{v}),$$

while the quasilinear theory does not specify a source for fluctuations [neither $\tilde{f}_{s1}(t=0)$, nor $\hat{\phi}(t=0)$].

(3) The kinetic theory uses a definition for ensemble average that integrates the initial particle positions

$$\langle\ldots\rangle\equiv\prod_{l=1}^{N}\int d^3x_{lo}d^3v_{lo}\frac{f_l(\mathbf{v}_{lo})}{(nV)^N}(\ldots),$$

while quasilinear theory uses a spatial average over the fluctuation scale-length

$$\langle\ldots\rangle=\frac{1}{V}\int d^3x\,(\ldots).$$

Difference (1) results in the neglect of the first and third terms of equation 3.73 in conventional quasilinear theory, because the initial $\tilde{f}_{s1}(t=0)$ is not accounted for by the series assumption for the timescale of $f_{s1}(\mathbf{x},\mathbf{v},t)$. It is, of course, kept when using Fourier-Laplace integral transforms such as the kinetic theory uses. In fact, in the kinetic theory, the first term of equation 3.73 becomes the drag term for individual collision operators. Difference (1) can easily be eliminated by using the equation 3.73, instead of equation 3.11, in the quasilinear collision operator. Is it then possible to capture individual collision operators within quasilinear theory? The answer is still no, because of differences (2) and (3).

In the kinetic theory, after inserting the discrete particle source for $\tilde{f}_{s1}(t=0)$, and applying the ensemble average to equation 3.73, all the terms from the homogeneous part of $\tilde{f}_{s1}(t=0)$ [that is the $(2\pi)^3\delta(\mathbf{k})f_{so}(\mathbf{k})$ part] vanish. Also, all of the "unlike" particle terms with $s' \neq s''$ vanish (this was shown in equation 2.28). Likewise, all of the third term of equation 3.73 vanishes. Here, the terms with $s \neq s''$ vanish after the ensemble average and the remaining $s''=s$ term vanishes because it has odd-parity in k-space. However, if the origin of fluctuations is not specified, and/or a spatial average is used instead of the kinetic ensemble average, one cannot show that the "unlike" particle correlation terms vanish; in this case, many additional nonzero terms are left beyond the kinetic theory result.

Unless the discrete particle source is specified, the result from averaging equation 3.73 cannot be split into individual component collision operators. This is because the first and second terms will depend on the product of field quantities, and the cross-term contributions to these ($s'' \neq s'$) will remain. These do not remain in the kinetic theory, allowing the result to be written in terms of a sum over individual species interactions [s with s', giving a total effect of $\sum_{s'} C(f_s, f_{s'})$]. That "unlike" particle correlations vanish is the essential difference between the kinetic and quasilinear formulations. Without specifying the origin of fluctuations, this identification cannot be made. This fundamental difference with the kinetic approach prevents conventional quasilinear theory from being written in terms of individual collision operators. The ability to do so is a distinctly kinetic result.

Chapter 4

Langmuir's Paradox

Irving Langmuir's most acclaimed contributions to science were for his foundational work in surface chemistry. He was awarded the 1932 Nobel Prize in Chemistry "for his discoveries and investigations in surface chemistry." But Langmuir had broad interests and he was also a founder of another field: plasma physics. That Langmuir was a true founder of plasma physics is evidenced by the fact that he named the state of matter in 1928 [70]. Although much of his published work concerned fundamental characteristics of plasma, Langmuir was an industrial physicist (at General Electric) and his work was motivated by a practical device: the gas-filled incandescent lamp.[1] In this chapter, we will study an anomaly that Langmuir discovered in his incandescent lamps (first reported in 1925 [13, 14]) that remains unresolved to this day. It is now referred to as Langmuir's paradox [15].

4.1 Introduction to Langmuir's Paradox

Langmuir's paradox is a measurement of anomalously fast equilibration of the electron distribution function to Maxwellian in low-temperature gas-discharge plasma. In 1925, Langmuir reported [13] that despite the fact that the shortest collision length for electron scattering (from binary Coulomb collisions, or collisions with neutrals) was much longer than the diameter of his plasma, the electron distribution was Maxwellian at all energies. This was a surprising result because sheaths near the plasma boundaries selectively remove high energy electrons. Without a means to scatter electrons to higher energy and replete the otherwise missing tail, it was inexplicable how the entire distribution remained Maxwellian. Thirty years later, Gabor named this phenomena Langmuir's paradox [15].

In section 4.1.1, we review the experimental devices and measurements used by Langmuir in his

[1] For a history and tribute to Langmuir's work in the foundation of plasma physics, see the February 2009 issue of *Plasma Sources Science and Technology*.

original work on this topic. Gabor's definition of Langmuir's paradox is discussed in section 4.1.2. Previous attempts to resolve this discrepancy (including Langmuir's own attempts as well as those of Gabor and others) are reviewed in section 4.1.3. Finally, in section 4.1.4, we show how the kinetic theory of instability-enhanced interactions developed in chapter 2 may be used to resolve Langmuir's paradox.

4.1.1 Langmuir's Seminal Measurements

A few years before introducing the term plasma, Langmuir studied electron scattering in low-pressure gas-discharges. In his 1925 *Physical Review* article "Scattering of Electrons in Ionized Gases" [13], Langmuir reported seminal measurements of the properties of electrons in these plasmas. His experimental apparatus was a spherical discharge tube approximately 3 cm in diameter, which was made of glass and energized by electrons emitted from a hot filament [13, 14]; see figure 4.1. He discovered that nearly all of the discharge was a quasi-neutral plasma, but that because electrons diffused to the boundaries much faster than the more massive positively charged ions (by about 400 times), a thin electric field, which he later named a sheath [70], surrounded the plasma and acted to reflect most of the incident electrons (about 399/400 of them) so that the electron and ion fluxes balanced at the boundary. The presence of sheaths allows a plasma to remain in quasi-netural in steady-state operation. For such an ambipolar ion sheath, the potential drops $e\Delta\phi_s \approx -T_e \ln\sqrt{2\pi m_e/M_i}$ ($\approx 5T_e$ for mercury) over a few electron Debye lengths $\lambda_{De} \equiv \sqrt{T_e/4\pi e n_e}$. In Langmuir's discharge, the electron temperature was $T_e \approx 2$ eV and the electron density was $n_e \approx 10^{11}$ cm^{-3}, so the sheath potential drop was $e\Delta\phi_s \approx 10$ eV, and the Debye length was $\lambda_{De} \approx 3 \times 10^{-3}$ cm. It was later shown that an additional, but much weaker, presheath electric field is also present in plasmas which accelerates the ion fluid speed to the sound speed, $V_i \geq c_s \equiv \sqrt{T_e/M_i}$ at the presheath-sheath boundary. This result is commonly attributed to Bohm [38] (it is called the Bohm criterion), but it was also appreciated in Langmuir's earlier works deriving the "plasma balance equation" [72]. The potential in the presheath of these discharges typically drops $e\Delta\phi_{ps} \approx T_e/2$ over a distance characteristic of the ion-neutral collision mean free path $\lambda^{i/n} \gg \lambda_{De}$, see figure 4.2.

In his apparatus, Langmuir measured the electron velocity distribution function (EVDF) using an

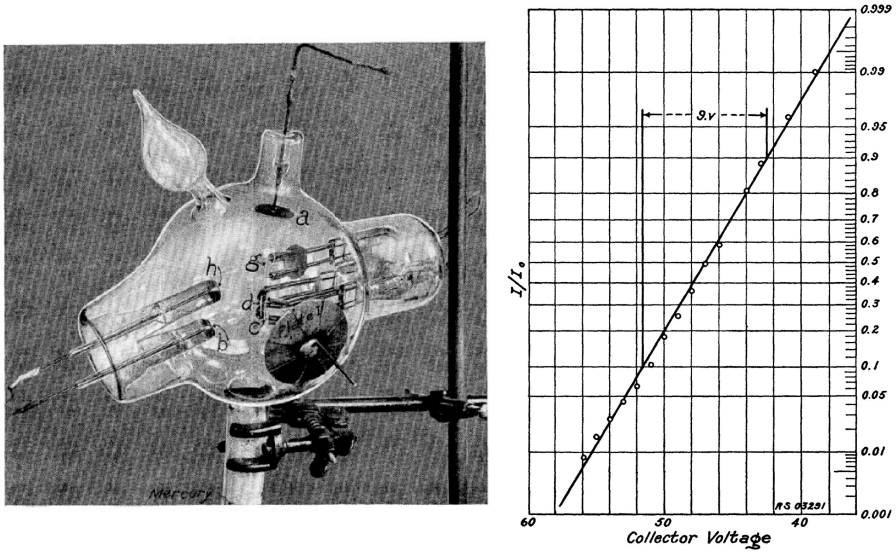

Figure 4.1: Photograph of Langmuir's discharge reproduced from reference [71], and one of his probe traces reproduced from reference [13]. Copyright 1929 and 1925 by The American Physical Society. This probe trace shows a Maxwellian distribution for energies in excess of 50 V, which is approximately five times the sheath energy.

electrostatic probe (now called a Langmuir probe). The Langmuir probe technique scans the bias on a conductor inserted into a plasma and can be used to infer the EVDF based on the measured current-voltage characteristic. A Maxwellian EVDF shows up as a straight line on a semi-log current-voltage trace; see figure 4.1. Surprisingly, Langmuir found that the EVDF was Maxwellian at all velocities despite the fact that his calculated electron-electron scattering length, $\lambda^{e/e}$, was much longer than the tube diameter. The electron-ion and electron-neutral scattering lengths where also much longer than the tube diameter. Langmuir thus expected significant depletion of the EVDF for electrons with energy large enough to escape the sheath; in particular electrons with $v_{\parallel} \gtrsim v_{\parallel c} \equiv \sqrt{2\Delta\phi_s/m_e}$. Here the $\parallel$ direction is parallel to the sheath electric field (perpendicular to the bounding surface). It was also unexplainable how his discharge could remain lit because the vast majority of ionization events were attributed to the very same electrons in the tail of the Maxwellian EVDF (rather than the filament emitted electrons) that basic scattering theory predicted to be missing. Langmuir's measurements implied that some anomalous mechanism for electron scattering must have been present which was capable of boosting the velocity of many electrons and rapidly establishing the Maxwellian equilibrium.

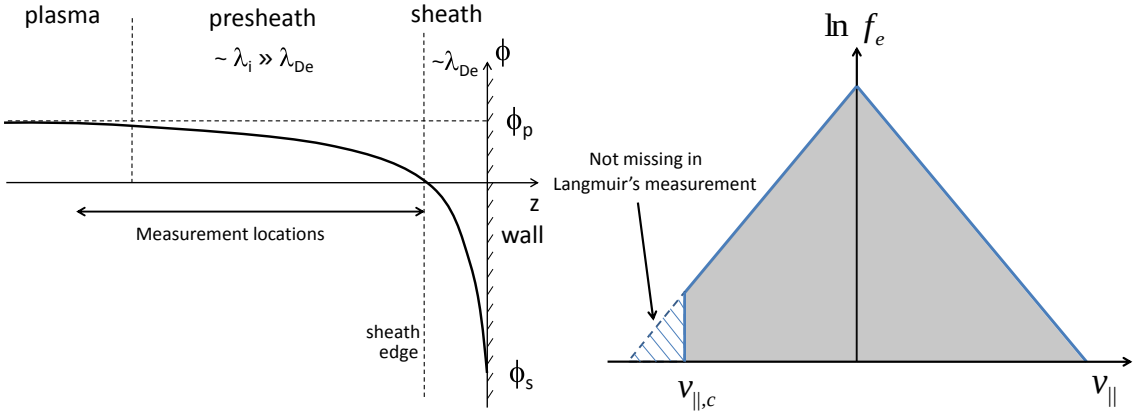

Figure 4.2: Sketches of the plasma-boundary transition region and the natural logarithm of a truncated Maxwellian distribution function. The truncation was expected, but not found in Langmuir's measurements. Instead a Maxwellian was measured at all velocities.

Langmuir summarized the major conclusions of his experiments on page 591 of reference [13]:

> This discussion of the results obtained from a study of low pressure arcs shows (1) that the free electrons have velocities with a Maxwell distribution corresponding (in the case of mercury vapor at 1 bar) to a temperature of 30,000°; (2) that this distribution is maintained even when the walls are negatively charged and hence are constantly removing the faster electrons; (3) that the number of collisions with atoms and electrons is far too small to maintain this distribution, the mean free path being of the order of the tube diameter; (4) that mobility experiments indicate that the electrons suffer at least ten changes of momentum between consecutive collisions with atoms.

The sheath potential drop in Langmuir's discharge was 15.5 volts, while the EVDF was measured to be Maxwellian out to energies in excess of 50 volts (see figure 4.1).

Langmuir goes on to describe a number of experiments where the plasma is generated by electrons emitted from a hot cathode, or a heated filament, with varying energies and currents (he calls these *primary electrons*). He finds that there is a connection between the energy input into the plasma from the primary electrons and the electron temperature in the bulk of the plasma (what he calls *ultimate electrons*), but that in all cases the EVDF continues to be Maxwellian to all diagnosable energies. The

total EVDF consisted of three regions in velocity-space that all had Maxwellian distributions, but with differing temperatures: the fast primary electrons, the thermal ultimate electrons and an intermediate region which he called *secondary electrons*. The density of ultimate electrons was over 1000 times the primary or secondary densities.

By turning his probe with respect to the beam, he could show that the primaries had a flow component in the direction that they where injected, but that enhanced scattering must have occurred because they had a Maxwellian distribution with temperature 1420°. Thus, the beam was not monoenergetic as expected and many primaries obtained energies greater than the energy at which they where injected. By varying the current through the hot filament (or cathode heating), Langmuir could change the density of the injected electron beam. He found that less dense beams had a colder temperature, indicative of more monenergetic beams. However, the Maxwellian property of the EVDF of ultimate electrons remained unchanged. From this series of experiments, Langmuir could conclude that the Maxwellian tail of the ultimate electron distribution was not simply being filled by slowing primary electrons, but was necessarily due to enhanced scattering of lower energy electrons in the thermal part of the distribution (amongst the ultimate electrons themselves). Enhanced electron scattering appeared to occur amongst the primary electrons themselves as well.

4.1.2 Gabor's Definition of Langmuir's Paradox

Despite Langmuir's emphasis on the importance of resolving this major discrepancy,[2] the problem was largely ignored by physicists. Shortly after Langmuir's publications, Dittmer repeated his experiments corroborating his results [73]. Aside from this work, there was very little immediate response to the problem. Perhaps the mysteries uncovered by Langmuir were overshadowed by the revolution in physics that was being brought about by quantum mechanics at the time (one must also remember that Langmuir's was a new field, and very few people were working on the physics of ionized gases in 1925). Whatever the reason for the inattention, it took thirty years before a serious second look was taken at Langmuir's measurements.

In 1955, Dennis Gabor and coworkers took up the issue of electron scattering uncovered by Langmuir. Rather than paraphrase their *Nature* article, we simply quote the section of the introduction in which

[2]For instance, Langmuir discussed this problem at the "International Congress on Physics" in Como, 1927 [14].

they introduce and define 'Langmuir's paradox'. After introducing Langmuir's theory of gas discharge plasmas, Gabor writes on page 916 of reference [15]:

> This theory was highly successful in all respects except one; but this exception, as Langmuir showed in a number of important papers [13, 14], presented one of the worst discrepancies known in science. Once the existence of electron temperatures was accepted, the theory of Langmuir and Mott-Smith could account for all observations, but this phenomenon itself remained not only unexplained but also inexplicable. Maxwellian energy distributions have been observed, in discharge tubes of a few centimetres in diameter, at mercury pressures so low that the mean free path was several times the tube diameter; and, moreover, at such small electron densities that it would take an electron, in the average, a path of the order of 10 m to exchange amounts of the order of the mean energy with other electrons. The tube wall eliminates all the time electrons above a certain energy limit, and an electron collides, on the average, some thousand times with the boundary sheath before it has a chance to pick up energies of this order. Yet probe measurements fail to reveal any deficit of fast electrons among those returning from the wall, even at a distance of only a few millimeters from it.
>
> Though the existence of electron temperatures has become a well-established part of our knowledge of low-pressure discharges, the fact that this phenomenon persists in a region in which it has theoretically no right to exist appears to have been largely ignored, in spite of Langmuir's emphasis. Our investigations, which started five years ago, have now led to the elucidation of what we propose to call 'Langmuir's paradox'.

Recall from Langmuir's paper [13] that he reported enhanced electron scattering for both the primary and ultimate electron species. One important note to take from Gabor's definition of Langmuir's paradox is that it refers to the fast filling of the tail of the ultimate electron distribution; as opposed to thermal spreading of the primary electron beam. Langmuir thought it likely that the two mechanisms where related, which is a point that we agree on, but we wish to be specific about what refer to as 'Langmuir's paradox.' For this we use Gabor's definition from the quote above. We stress this point because, although essentially everyone who has looked at this problem since Gabor has used his

definition, a popular introductory plasma physics textbook [74] has recently called the fast slowing of an injected electron beam in a plasma 'Langmuir's paradox.' This is not what Gabor defined as 'Langmuir's paradox'; the important differences being that Gabor's definition refers to the ultimate electrons (although similar mechanisms may scatter primaries) and, more importantly, that it is specific to the distribution becoming uniquely Maxwellian (not simply that the injected electrons slow down quickly).

4.1.3 Previous Approaches to Resolve Langmuir's Paradox

Plasma kinetic theory has evolved significantly in the 85 years since Langmuir published his measurements, so one should first ask if modern plasma kinetic theory predicts a shorter electron-electron collision length that can explain Langmuir's paradox. The answer to this, as we will show in section 4.4.1, is no. If one uses the Lenard-Balescu or Landau equation to calculate the electron-electron collision length, on finds a prediction of 28 cm for Langmuir's plasma. This is essentially the same as the 30 cm length quoted by Langmuir in his paper (an impressive estimate for the time!). Thus, even using modern plasma kinetic theory, conventional Coulomb interactions in a stable plasma cannot explain the measurements; some other mechanism must be identified in order to explain Langmuir's paradox.

In his original publication on the topic, Langmuir speculated as to what this physical mechanism might be [13]. One suspicion was that "radiation quanta" (which we now call photons) emitted from excited gas molecules might be responsible. However, providing a quick estimate of this effect Langmuir shows it to be insufficient in scattering electrons by a factor of 10^{16}. He then speculates that a 'resonance radiation' processes whereby the three body interaction of excited neutral atoms, radiation quanta and an electron could significantly enhance the effect. However, he cites an experiment done in higher vacuum where the neutral density was too low for this effect to work, but in which case the electron distribution remained Maxwellian. Thus, scattering by radiation was not able to explain the measurements. On the last page of his paper, Langmuir suggested that 'oscillations' in the plasma may be responsible for the electron scattering. We will concentrate on unstable waves as an explanation in this work, which is similar to what Langmuir calls 'oscillations'. However, Langmuir appears to have dismissed this as a possible cause because he failed to measure any oscillations when using a radio

detector capable of detecting oscillations of 0.5 m wave-length (the oscillations we will consider have wavelengths less than 1 mm, which would not be detected by Langmuir's radio wave detector). Dittmer [73] supported the idea that waves where responsible for the scattering but, using a similar technique as Langmuir, also failed to measure them.

Another early thought was that circuit oscillations may be present within the electronics of the collecting probe, such that it is only biased more positive than the plasma in a time averaged sense, and that the actual electron energies are not accurately represented [75]. However, Langmuir shows that such oscillations would have to be at least 40 V to explain the measurements, and could show that, if present at all, oscillations in his circuitry where less than 1 V [14].

After a thirty year lull in research on the topic, Gabor named 'Langmuir's paradox' and also took a closer look at the idea that waves might be present in the plasma which could scatter electrons [15]. Plasma physics as a subject had progressed rapidly in these thirty years (but not Langmuir's electron scattering problem) and by '55 the subject of plasma waves, in particular, had developed significantly. By that time, Landau damping [4] had been developed and it was generally appreciated, at least within fluid theory, that waves could be unstable and grow to larger amplitudes in a plasma. Furthermore, it was well known that plasma waves could be of very short wavelength and very high frequency; two things preventing Langmuir's early attempts to measure them.

Gabor *et al* [15] used an oscillogram technique to measure waves in the plasma. Although the technique was rudimentary (by today's standards), they did find waves in the MHz frequency range near the plasma boundaries. They did not know the source of the waves at the time, but thought that they might be responsible for enhancing the electron scattering. However, there was no theory available at the time to either describe the wave dispersion or describe to what order the waves are important in enhancing electron scattering. We propose that Gabor's hypothesis is right, and in the next section we provide the theory that was missing in his day to support it (the instabilities are of the ion-acoustic type which fluctuate in the MHz range, consistent with his measurement). Unfortunately, after Gabor's reconsideration of this problem attention to it again died down and only a few scattered references have been added to the literature since.

One of these was a paper by Gorgoraky [76] that considered the role of electron-neutral collisions. He investigated the possibility that the electronic polarizability of neutral atoms and molecules can

affect the ionization rate from electron collisions with neutrals. If the neutral gas was largely in excited (metastable) states, it may lead to significantly larger ionization frequencies than are otherwise predicted. However, he did not apply this theory to Langmuir's plasma, and the work has not been followed up on. Hence, it is difficult to judge the relevance of this theory to Langmuir's paradox. Langmuir's measurements showed that the formation of the Maxwellian electron distribution had essentially no dependence on the neutral pressure. This seems contradictory to any proposal that suggests electron-neutral collisions cause the electron distribution to become Maxwellian at a temperature much larger than the neutral temperature.

Other modern work on Langmuir's paradox has been published by Tsendin [77, 78]. Tsendin has developed a non-local approximation to electron kinetics that is based on separating the distribution function into a lowest-order component that has a speed, rather than velocity, dependence and a next-order component in which the spatial parts of the velocity dependence are expanded in spherical harmonics: $f(\mathbf{x}, \mathbf{v}, t) = f_0(\mathbf{x}, v, t) + f_1(\mathbf{x}, v, t)Y_1^m(\vartheta, \varphi)$, in which $f_0 \gg f_1$. He uses this approximation to determine the distribution function of a plasma; not to develop a collision operator. There is no enhanced electron scattering in this theory. Using his approximation, Tsendin calculates distribution functions that are essentially Maxwellian, and he claims that Langmuir's paradox may not exist. However, the theory provides no new mechanism for electron scattering, and it seems that his prediction of essentially Maxwellian distributions may be tied to the assumption above.

Previous attempts to solve Langmuir's paradox span a wide range of different ideas. These include approaches searching for an enhanced scattering mechanism, such as scattering by photons (with and without resonant interactions), by excited neutral atoms, or by waves, as well as approaches that suggest it may not really exist, such as circuit oscillations interfering with the measurements and the non-local approximation to electron kinetics. None of these previous theories have provided a definitive answer to Langmuir's paradox.

4.1.4 Our Approach to Resolve Langmuir's Paradox

In the rest of this chapter, we consider details of the plasma-boundary transition region and show that an instability-enhanced collective response (and hence fluctuations), due to ion-acoustic instabilities

causes electron-electron scattering to occur much more frequently than it does by Coulomb interactions alone. This collective response arises from the instability amplification of discrete particle motion; it effectively extends the range over which particles interact beyond the Debye sphere in which Coulomb interactions are confined. Since these ion-acoustic instabilities convect out of the plasma before reaching nonlinear levels, turbulence theories are not applicable in describing the enhanced electron scattering. We proved in section 3.4 that the Boltzmann $\mathcal{H}$-theorem remains valid when this collective response is present and that Maxwellian is the unique equilibrium. We will show that this is established at a rate rapid enough to be consistent with Langmuir's measurements.

This approach builds on the suggestions by Langmuir [13], Dittmer [73], Gabor [15], and others [79] that wave-particle scattering by instabilities near the discharge boundaries may be responsible for enhanced electron scattering. Gabor's previous experimental work gave compelling evidence for the presence of waves in the MHz frequency range near sheaths [15]. However, these previous works could not provide a definitive answer to Langmuir's paradox because there was no theory to describe either the wave dispersion or a kinetic theory to describe how the resultant wave-particle scattering leads to the rapid establishment of a uniquely Maxwellian distribution. We propose that the waves Gabor measured are ion-acoustic instabilities, which are in the MHz range. To show that they lead to a Maxwellian, we employ the kinetic theory of unstable plasmas that was developed in chapter 2. This theory is well suited to describing the effects of ion-acoustic instabilities in the presheath because they convect out of the plasma while still in a linear growth regime. The prediction that the waves Gabor measured are ion-acoustic instabilities is also supported by the fact that the measured waves had the correct frequency and that they were only present near the boundaries.

Other instabilities may also have been present in Langmuir's discharge, which could lead to instability-enhanced collisional effects. For example, the primary electrons emitted from filaments may cause a bump-on-tail instability. In fact, this is a probable mechanism for the observation of enhanced scattering amongst the primaries and it is a problem that can be solved using the approach presented in this chapter. Two-stream instabilities may also be present in presheaths when there are multiple species of ions, or if some ions are multiply-charged; this instability will be studied in detail in chapter 6 with emphasis on how it affects the ions. We are primarily interested in ion-acoustic instabilities because they are universal in presheaths of gas-discharges with low temperature and pressure. Thus, they apply

to all discharges susceptible to Langmuir's paradox regardless of the mechanism by which the plasma is generated.

We will apply our kinetic theory to the same discharge parameters that Langmuir used in his original experiments [13]. This was a mercury plasma with electron (plasma) density $n_e \approx 10^{11}$ cm^{-3}, neutral density $\approx 10^{13}$ cm^{-3} (0.3 mTorr), and ion and electron temperatures of $T_i \approx 0.03$ eV and $T_e \approx 2$ eV respectively.

4.2 The Plasma-Boundary Transition: Sheath and Presheath

In order to apply our theory, we must determine the equilibrium ion flow velocity throughout the plasma-boundary transition region. This is required because it determines the dielectric function and dispersion relation for the ion-acoustic instabilities, which in turn determines the instability-enhanced collision operator and hence the resulting collision frequency. The typical plasma-boundary transition region (one in which equal electron and ion fluxes reach the boundary surface) can be split into two regions: sheath and presheath. We will see that most of the electrostatic potential drop is in the very thin sheath, and that the potential drop of the presheath is comparatively less, but it extends much farther into the plasma. We will be primarily interested in the presheath because, although the instability growth rate is smaller there than in the sheath, it is sufficiently long that the unstable waves have time to grow to significant amplitudes. In this section, we will follow the presentation of Lieberman and Lichtenberg [80] to review basic sheath equilibrium.

4.2.1 Collisionless Sheath

The simplest ion sheath model is the so called "collisionless sheath." It assumes (1) Maxwellian electrons with a temperature T_e, (2) one species of cold ions, $T_i = 0$ and (3) quasineutrality at the "plasma-sheath interface": $n_e(0) = n_i(0)$. Here $z = 0$ is the location of the plasma-sheath interface. The plasma-sheath interface is the location where quasineutrality breaks down. We will be more rigorous in defining it's location later. We also assume a 1-D problem where the electric field and ion fluid flow are perpendicular to the material boundary.

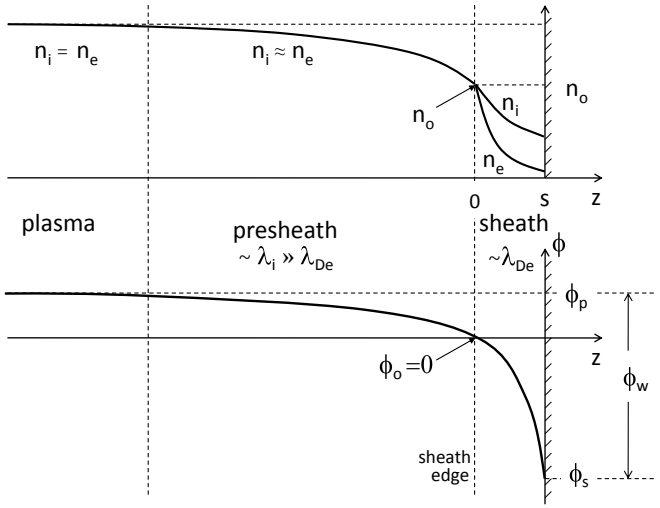

Figure 4.3: Schematic drawing of the ion and electron density profiles and the electrostatic potential profile throughout the plasma-boundary transition region; including plasma, presheath and sheath. The electrostatic potential is referenced to zero at the sheath edge. Figure adapted from Lieberman and Lichtenberg [80].

If ions are collisionless in the sheath, their dynamics are described by energy conservation

$$\frac{1}{2}M_i V^2(z) = \frac{1}{2}M_i V_o^2 - q_i\phi(z) \tag{4.1}$$

where $V(z)$ is the ion fluid speed, V_o is the ion fluid speed at the sheath edge and $\phi(z)$ is the electric potential. We take the convention that $\phi(z=0)=0$ and is negative in the sheath in accordance with figure 4.3. The continuity of ion flux, assuming no ionization, recombination, or any other sort of charge transfer in the narrow sheath is

$$n_i(z)V(z) = n_{io}V_o. \tag{4.2}$$

Equations 4.1 and 4.2 combine to give an expression for the ion density as a function of position

$$n_i(z) = n_{io}\left(1 - \frac{2q_i\phi(z)}{M_i V_o^2}\right)^{-1/2}. \tag{4.3}$$

Maxwellian electrons obey the Boltzmann relation in the absence of collisions

$$n_e(z) = n_{eo}\exp\left(\frac{e\phi(z)}{T_e}\right). \tag{4.4}$$

Due to quasineutrality at the sheath edge, $n_{eo} \approx n_{io} \equiv n_o$. Substituting n_i and n_e into Poisson's equation gives

$$\frac{d^2\phi}{dz^2} = 4\pi\big(en_e - q_i n_i\big) = 4\pi n_o\left[e\exp\left(\frac{e\phi(z)}{T_e}\right) - q_i\left(1 - \frac{\phi(z)}{\mathcal{E}_o}\right)^{-1/2}\right] \tag{4.5}$$

where $q_i\mathcal{E}_o \equiv \frac{1}{2}M_iV_o^2$ is energy of the ion fluid flow at the sheath edge.

In principle, equation 4.5 can be solved for ϕ subject to boundary conditions on ϕ that come from considerations of total current balance for the plasma. However, one of these conditions is implicit in equation 4.5 itself; this is the Bohm criterion.

4.2.2 The Bohm Criterion

Multiplying equation 4.5 by $d\phi/dz$ and integrating over z gives

$$\int_0^\phi \frac{d\phi}{dz}\frac{d}{dz}\left(\frac{d\phi}{dz}\right)dz = 4\pi n_s\int_o^\phi \frac{d\phi}{dz}\left[e\exp\left(\frac{\phi}{T_e}\right) - q_i\left(1 - \frac{\phi}{\mathcal{E}_o}\right)^{-1/2}\right]dz, \tag{4.6}$$

which upon evaluating the integrals yields

$$\frac{1}{2}\left(\frac{d\phi}{dz}\right)^2 = 4\pi n_o\left[T_e\exp\left(\frac{\phi}{T_e}\right) - T_e + 2q_i\mathcal{E}_o\left(1 - \frac{\phi}{\mathcal{E}_o}\right)^{1/2} - 2q_i\mathcal{E}_o\right] \tag{4.7}$$

where $\phi(z=0) = 0$ and $d\phi/dz = 0$ at $z = 0$.

Since the left hand side of equation 4.7 is positive, so is the right side. This means that the electron density must always be less than the ion density in the sheath. Expanding the right side in a series for small ϕ and assuming that $q_i = e$ yields the condition

$$\frac{1}{2}\frac{\phi^2}{T_e} - \frac{1}{4}\frac{\phi^2}{\mathcal{E}_o} \geq 0. \tag{4.8}$$

This is satisfied if $\mathcal{E}_o \geq T_e/2$, which is equivalent to

$$\boxed{V_o \geq c_s \equiv \sqrt{\frac{T_e}{M_i}}} \tag{4.9}$$

which is the **Bohm criterion** [38] for a single ion species plasma. Equality typically holds in equation 4.9, providing a boundary condition for the ion flow at the plasma edge [81].

4.2.3 The Child Law and Sheath Thickness

Assuming that the initial ion energy at the sheath edge $\mathcal{E}_o$ is much smaller than the electric potential energy in the sheath $e\phi$, the ion energy and conservation equations are

$$\frac{1}{2}M_i V^2(z) = -q_i\phi(z) \quad \text{and} \quad q_e n_i(z)V(z) = J_o \tag{4.10}$$

where $J_o \equiv q_i n_o c_s$ is the constant ion current entering the sheath, in which equation 4.9 has been applied, and the electron density in the sheath is assumed to be negligible $n_e = 0$. Solving for the ion density

$$n_i(z) = \frac{J_o}{q_i}\left(-\frac{2q_i\phi}{M_i}\right)^{-1/2} \tag{4.11}$$

and using this in Poisson's equation (with the assumption $n_i \gg n_e$ in the sheath), we find

$$\frac{d^2\phi}{dz^2} = -4\pi J_o\left(-\frac{2q_i\phi}{M_i}\right)^{-1/2}. \tag{4.12}$$

Multiplying by $d\phi/dz$ and integrating from 0 to z gives

$$\frac{1}{2}\left(\frac{d\phi}{dz}\right)^2 = 8\pi J_o\left(\frac{2q_i}{M_i}\right)^{-1/2}\sqrt{-\phi} \tag{4.13}$$

where we have again used $d\phi/dz = \phi = 0$ at $z = 0$. Taking the square root and integrating again gives

$$(-\phi)^{3/4} = 3\sqrt{\pi J_o}\left(\frac{2q_i}{M_i}\right)^{-1/4} z. \tag{4.14}$$

Setting the potential at the boundary surface $z = s$ to be $\phi = -\phi_s$ and solving for J_o gives

$$J_o = \frac{1}{9\pi}\sqrt{\frac{2q_i}{M_i}}\frac{\phi_s^{3/2}}{s^2} \tag{4.15}$$

which is the well-known Child law. Using the Bohm criterion $J_o = q_i n_o c_s$, and again applying the assumption $q_i = e$, this gives an expression for the sheath thickness

$$\boxed{s = \frac{\sqrt{2}}{3}\lambda_{De,o}\left(\frac{2e\phi_s}{T_e}\right)^{3/4},} \tag{4.16}$$

in which $\lambda_{De,o} \equiv \sqrt{T_e/(4\pi e n_o)}$ is the electron Debye length at the sheath edge.

Solving equations 4.14 and 4.15 for the potential profile in the sheath yields

$$\phi = -\phi_s\left(\frac{z}{s}\right)^{4/3}, \tag{4.17}$$

which implies the electric field is

$$E = \frac{4}{3}\frac{\phi_s}{s}\left(\frac{z}{s}\right)^{1/3}. \tag{4.18}$$

Finally, we can use the conservation of energy equation 4.10, along with the potential profile in equation 4.17, to determine the ion flow speed profile in the sheath. Applying Bohm's boundary condition that ions enter with a flow speed equal to the sound speed [$V(z=0) = c_s$], these yield

$$\boxed{V(z) = c_s + \sqrt{\frac{2e\phi_s}{M_i}}\left(\frac{z}{s}\right)^{3/2}.} \tag{4.19}$$

Recall that for Langmuir's mercury discharge, in which the boundaries collect equal electron and ion current, the wall charge to $e\phi_s \approx 5T_e$. Equation 4.16 shows that in Lanmguir's discharge, the sheath length is $s \approx 2.6\,\lambda_{De,s}$, and the ion velocity profile ranges from the sound speed c_s at the sheath edge to approximately $V_s = 4.5\,c_s$ at the boundary surface. Thus, ions are accelerated to high speeds over a sheath thickness of a few Debye lengths.

4.2.4 The Presheath

Next, we consider the presheath. The presheath separates the highly non-neutral sheath and the quasineutral plasma. It acts to accelerate ions from a fluid flow speed of essentially zero in the plasma to the ion sound speed at the sheath-presheath interface. To model the presheath, we will use a 1-D modified mobility-limited flow model due to Riemann [81]. For ions, this model uses the one dimensional steady-state fluid continuity equation

$$\frac{d}{dz}\left(n_i V_i\right) = 0, \tag{4.20}$$

and momentum equation

$$V_i\frac{dV_i}{dz} = \frac{q_i}{M_i}E - \nu_c V_i, \tag{4.21}$$

along with Poisson's equation

$$\frac{d^2\phi}{dz^2} = -4\pi e(n_i - n_e). \tag{4.22}$$

Reimann's model assumes that the electron density obeys the Boltzmann relation $n_e = n_o \exp(e\phi/T_e)$. Here ν_c is the ion-neutral collision frequency that will be shown to determine the presheath length scale. The ion-neutral collision frequency can be set by elastic ion-neutral collisions, ionization collisions or charge exchange collisions. In the plasmas of interest, charge exchange collisions typically dominate.

The Boltzmann relation for electrons, along with the assumption of quasineutrality in the presheath, implies that for both species

$$\frac{dn}{dz} = \frac{en}{T_e}\frac{d\phi}{dz} = -\frac{e}{T_e}En \tag{4.23}$$

which, when inserted into the continuity equation 4.20, gives

$$\frac{dV_i}{dz} = \frac{e}{T_e}EV_i. \tag{4.24}$$

Putting equation 4.24 into equation 4.21 and assuming singly charged ions ($q_i = e$), gives

$$V_i = \mu E\left(1 - \frac{V_i^2}{c_s^2}\right) \tag{4.25}$$

in which

$$\mu \equiv \frac{e}{M_i\nu_c} \tag{4.26}$$

is the "mobility." Putting n_i from the continuity equation 4.20, with the boundary condition provided by Bohm's criterion $n_iV_i = n_oc_s$ (where n_o is the density at the sheath edge) along with n_e from the Boltzmann relation into Poisson's equation 4.22, yields

$$\frac{d^2\phi}{dz^2} = -4\pi e\left(n_o\frac{c_s}{V_i} - n_oe^{e\phi/T_e}\right) \quad \Rightarrow \quad \left(\frac{c_s}{V_i} - e^{e\phi/T_e}\right) = \left(\frac{\lambda_{De_o}}{l}\right)^2\frac{d^2(-\Phi)}{dZ^2}, \tag{4.27}$$

In which $\Phi \equiv e\phi/T_e$ and $Z \equiv z/l$ where l is the presheath length scale. Quasineutrality comes from the smallness of $\lambda_{De,o}/l$ where $l \approx \lambda^{i/n}$ (the dominant ion collision length) in the presheath.

Assuming $\lambda_D/l \to 0$, equation 4.27 reduces to

$$\phi = \frac{T_e}{e}\ln\left(\frac{c_s}{V_i}\right) \tag{4.28}$$

which is a quasineutrality relation. This gives an expression for the electric field

$$E = -\frac{d\phi}{dz} = \frac{T_e}{e}\frac{1}{V_i}\frac{dV_i}{dz} \tag{4.29}$$

which, when inserted into the momentum equation 4.21, or equivalently equation 4.25, gives the expression

$$\left(\frac{c_s^2 - V_i^2}{V_i^2\nu_c}\right)dV_i = dz. \tag{4.30}$$

Equations 4.28 and 4.30 determine $V_i(\phi)$ and $z(V_i)$. The latter can then be solved for $\phi(z)$, which produces $E(z)$. Equation 4.25 is then used to calculate $V_i(z)$.

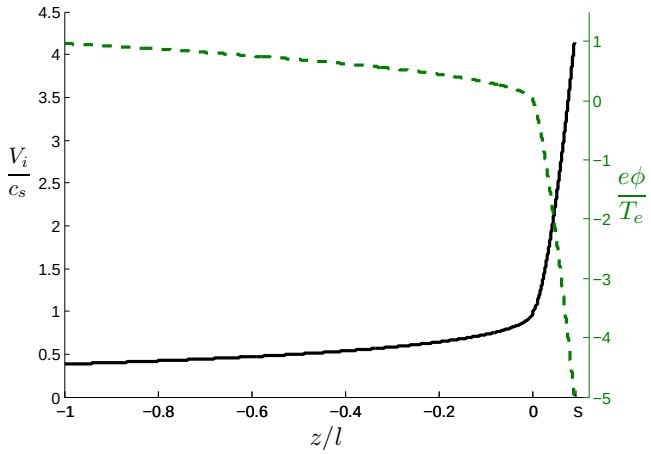

Figure 4.4: Profiles of the ion fluid speed (normalized to c_s and denoted by the solid black line) and the plasma potential (normalized to T_e and denoted by the dashed green line) throughout the presheath and sheath. Here we have used the constant ion-neutral collision frequency solution for the presheath from equations 4.33 and 4.34. For the sheath portion, we have used equations 4.17 4.19 with $e\phi_s = 5T_e$, and have taken for the plots $l/\lambda_{De} = 30$. For Langmuir's plasma $l/\lambda_{De} \approx 10^4$, but we have taken an artificially small value so that the sheath can be resolved on the plot. In reality, the sheath is so narrow compared to the presheath that the speed and potential profiles would show up as nearly vertical lines on these plots.

This modified mobility-limited flow model has been verified experimentally to a distance $z \approx 2l$ [84, 85], beyond which the ion flow speed transitions to zero in the bulk plasma. We will not be concerned with this transition here because the ion flow is so small in this region that ion-acoustic instabilities are not excited. Two different cases of equation 4.30 are commonly considered: constant mean free path for ion-neutral collisions $\lambda^{i/n} = l$, $\nu^{i/n} = V_i/l$, or a constant collision frequency $\nu^{i/n} \approx c_s/l$. For each, analytic solutions to equations 4.28 and 4.30 can be obtained. For the $\lambda^{i/n} = \text{const}$ case, we find that the fluid speed profile is

$$\boxed{V_i(z) = c_s \exp\left\{\frac{1}{2} - \frac{z}{l} + \frac{1}{2}W\left[-\exp\left(2\frac{z}{l} - 1\right)\right]\right\}} \tag{4.31}$$

where W is the Lambert W function. The potential profile is

$$\frac{e\phi}{T_e} = \frac{z}{l} - \frac{1}{2} - \frac{1}{2}W\left[-\exp\left(2\frac{z}{l} - 1\right)\right]. \tag{4.32}$$

For the $\nu_c^{i/n} = \text{const}$ case, the flow speed is given by

$$\boxed{V_i(z) = c_s\left[1 - \frac{z}{2l}\left(1 - \sqrt{1 - 4l/z}\right)\right]} \tag{4.33}$$

and the potential profile is given by

$$\frac{e\phi}{T_e} = \operatorname{arccosh}\left(1 - \frac{z}{2l}\right). \tag{4.34}$$

In the remainder of this chapter, we will apply the constant collision frequency model from equations 4.33 and 4.34. The ion flow speed and potential profiles are not sensitive to which model is used for ion-neutral collisions (for the purposes of this work, it makes no difference which model is used). A plot of the ion flow speed and potential profiles throughout the plasma-boundary transition region are shown in figure 4.4. For the sheath portion of these plots, we have used equations 4.17 and 4.19. For the presheath portion, we have used the constant collision frequency model from equations 4.33 and 4.34. The length of the sheath is also exaggerated in the figure in order to resolve its features. To do so, we chose a value of $l/\lambda_{De} = 30$, whereas in Langmuir's plasma it is really $l/\lambda_{De} = 10^4$. If a realistic value for l/λ_{De} where used, the sheath would be so narrow relative to the presheath that it could not be distinguished on the plot in figure 4.4.

4.3 Ion-Acoustic Instabilities in the Presheath

Next, we consider instabilities that might be excited due to the ion flow in the plasma-boundary transition (in particular, the presheath). Recall from equation 2.18 that the electrostatic plasma dielectric function is

$$\hat{\varepsilon}(\mathbf{k},\omega) = 1 + \sum_s \frac{4\pi q_s^2}{k^2 m_s} \int d^3v \frac{\mathbf{k}\cdot\partial f_s/\partial \mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}}.$$

In order to evaluate the plasma dielectric function, the distribution functions for ions and electrons must be specified. We assume that ions have a Maxwellian distribution

$$f_i(\mathbf{v}) = \frac{n_i}{\pi^{3/2} v_{Ti}^3} \exp\left[-\frac{(\mathbf{v}-\mathbf{V}_i)^2}{v_{Ti}^2}\right]. \tag{4.35}$$

with flow speed V_i in the $\hat{z}$ direction: $\mathbf{V}_i = V_i\hat{z}$.

Using the coordinates (χ,η,ζ), aligned such that $\mathbf{k} = k\hat{\zeta}$, and applying the variable substitution $\mathbf{u} = \mathbf{v} - \mathbf{V}_i$, the ion term in the plasma dielectric function can be written

$$\int d^3v \frac{\mathbf{k}\cdot\partial f_i/\partial\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}} = \frac{n_i}{\pi^{3/2}v_{Ti}^3} \underbrace{\int_{-\infty}^{\infty} du_\chi \exp\left(-\frac{u_\chi^2}{v_{Ti}^2}\right)}_{=\sqrt{\pi}v_{Ti}} \underbrace{\int_{-\infty}^{\infty} du_\eta \exp\left(-\frac{u_\eta^2}{v_{Ti}^2}\right)}_{=\sqrt{\pi}v_{Ti}} \int_{-\infty}^{\infty} du_\zeta \frac{k\frac{d}{du_\zeta}\exp\left(-u_\zeta^2/v_{Ti}^2\right)}{(\omega - \mathbf{k}\cdot\mathbf{V}_i) - kv_\zeta}. \tag{4.36}$$

Applying the definitions $t \equiv u_\zeta/v_{Ti}$ and $w \equiv (\omega - \mathbf{k}\cdot\mathbf{V}_i)/v_{Ti}$, we find

$$\frac{4\pi q_i^2}{k^2 m_i}\int d^3v \frac{\mathbf{k}\cdot\partial f_i/\partial\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}} = \frac{\omega_{pi}^2}{k^2 v_{Ti}^2}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dt \frac{\frac{d}{dt}e^{-t^2}}{w-t} \tag{4.37}$$

in which $\omega_{pi}^2 \equiv 4\pi q_i^2 n_i/m_i$ is the ion plasma frequency. The integral in equation 4.37 can be written in terms of the plasma dispersion function

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dt \frac{\frac{d}{dt}e^{-t^2}}{w-t} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dt\left[\frac{d}{dt}\left(\frac{e^{-t^2}}{w-t}\right) - \frac{e^{-t^2}}{(w-t)^2}\right] = -\frac{d}{dw}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dt \frac{e^{-t^2}}{t-w} = -Z'(w). \tag{4.38}$$

Thus, the ion term of the plasma dispersion function can be written

$$\frac{4\pi q_i^2}{k^2 m_i}\int d^3v \frac{\mathbf{k}\cdot\partial f_i/\partial\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}} = -\frac{\omega_{pi}^2}{k^2 v_{Ti}^2} Z'\left(\frac{\omega - \mathbf{k}\cdot\mathbf{V}_i}{kv_{Ti}}\right). \tag{4.39}$$

Recall that the plasma dispersion function is defined as [82]

$$Z(w) \equiv \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dt \frac{e^{-t^2}}{t-w}. \tag{4.40}$$

Properties of the plasma dispersion function, and a generalization of it, are discussed in appendix C.

Next, we need to consider the electron term of the plasma dielectric function. For this Langmuir's paradox problem, we are trying to find a length scale over which the electron distribution function becomes Maxwellian. Initially, the electrons are expected to deviate from Maxwellian in that the tail of the electron distribution is missing. That is, the electron distribution function is expected to be a stationary Maxwellian, except that it is truncated in the velocity direction parallel to the sheath electric field for $v_{\parallel} \geq v_{\parallel c} = \sqrt{2\Delta\phi_s/m_e}$ in which $\Delta\phi_s$ is the sheath potential drop. Recall that for the ambipolar ion sheath of interest here $e\Delta\phi_s = -T_e \ln\sqrt{2\pi m_e/M_i}$, which was ≈ 10 eV in Langmuir's plasma [13]. The pertinent question here is: how does this truncated tail of the electron distribution function affect ion acoustic instabilities in the presheath?

The answer to this question is that the truncated tail has essentially no affect on the ion acoustic instabilities as long as the critical speed where the truncation occurs is faster than the electron thermal speed $v_{\parallel c} \gg v_{Te}$. A rigorous analysis of this, which takes into account the truncated Maxwellian distribution, is presented in appendix C.3. As appendix C.3 shows, accounting for the truncation of the Maxwellian distribution requires evaluation of a modified plasma dispersion function, called the incomplete plasma dispersion function. The incomplete plasma dispersion function is defined as [83]

$$Z(\nu, w) = \frac{1}{\sqrt{\pi}} \int_{\nu}^{\infty} dt \frac{e^{-t^2}}{t - w}. \tag{4.41}$$

For the truncated electron distribution function of interest here, $\nu = -v_{\parallel c}/v_{Te}$. Appendix C.3 shows that corrections to the usual plasma dispersion function representation for the electron term

$$\frac{4\pi e^2}{k^2 m_e} \int d^3v \frac{\mathbf{k} \cdot \partial f_e/\partial \mathbf{v}}{\omega - \mathbf{k} \cdot \mathbf{v}} = -\frac{\omega_{pe}^2}{k^2 v_{Te}^2} Z'\left(\frac{\omega}{k v_{Te}}\right), \tag{4.42}$$

which are rigorously accounted for using the incomplete plasma dispersion function are of the order $\mathcal{O}\big[\exp(-v_{\parallel c}^2/v_{Te}^2)(v_{Te}/v_{\parallel c})\big]$.

For the ambipolar ion sheath of interest here, and assuming that the ions are mercury,

$$\frac{v_{\parallel c}^2}{v_{Te}^2} = \frac{1}{v_{Te}^2}\frac{2\Delta\phi_s}{m_e} = -\frac{1}{v_{Te}^2}\frac{2T_e}{m_e}\ln\left(\frac{2\pi m_e}{M_i}\right) = -\ln\left(\frac{2\pi m_e}{M_i}\right) \approx 11. \tag{4.43}$$

Thus, we find that the magnitude of corrections due to the truncated electron tail are

$$\mathcal{O}\left[\frac{v_{Te}}{v_{\parallel c}} \exp\left(-\frac{v_{\parallel c}^2}{v_{Te}^2}\right)\right] \approx \sqrt{11} e^{-11} = 5 \times 10^{-5}, \tag{4.44}$$

which is entirely negligible. Thus, we can confidently write the full plasma dielectric function in the conventional plasma dispersion function dependent form

$$\hat{\varepsilon}(\mathbf{k},\omega) = 1 - \frac{\omega_{pi}^2}{k^2 v_{Ti}^2} Z'\left(\frac{\omega - \mathbf{k}\cdot\mathbf{V}_i}{k v_{Ti}}\right) - \frac{\omega_{pe}^2}{k^2 v_{Te}^2} Z'\left(\frac{\omega}{k v_{Te}}\right). \tag{4.45}$$

Since we are considering ion acoustic instabilities, we apply the assumptions $kv_{Ti} \ll \omega - \mathbf{k}\cdot\mathbf{V}_i$ and $\omega \ll kv_{Te}$. Recall that the small argument expansion for $Z'(w)$ is [82]

$$Z'(w) = -2iw\sqrt{\pi}e^{-w^2} - 2\big(1 - 2w^2 + \ldots\big) \quad \text{for } |w| \ll 1 \tag{4.46}$$

and that the asymptotic expansion for large argument is

$$Z'(w) \sim -2i\sigma w\sqrt{\pi}e^{-w^2} + \frac{1}{w^2}\left(1 + \frac{3}{2w^2} + \ldots\right) \quad \text{for } |w| \gg 1 \tag{4.47}$$

in which

$$\sigma \equiv \begin{cases} 0\ , & \Im\{w\} > 0 \\ 1\ , & \Im\{w\} = 0 \\ 2\ , & \Im\{w\} < 0 \end{cases} . \tag{4.48}$$

Equations 4.46 and 4.47 can be derived by applying integration by parts to the integral form of Z', as shown in appendix C.1. Finally, applying the small argument expansion to the electron term, and the large argument expansion to the ion term in equation 4.45, yields

$$\boxed{\hat{\varepsilon}(\mathbf{k},\omega) = 1 + \frac{1}{k^2\lambda_{De}^2} - \frac{\omega_{pi}^2}{(\omega - \mathbf{k}\cdot\mathbf{V}_i)^2} + i\frac{\sqrt{\pi}}{k^2\lambda_{De}^2}\frac{\omega}{kv_{Te}}.} \tag{4.49}$$

Here we have also anticipated that $\omega \sim kc_s$, so that $\exp(-\omega/kv_{Te}) \sim \exp(-\sqrt{m_e/M_i}) \approx 1$, and $\exp[-(\omega - \mathbf{k}\cdot\mathbf{V}_i)/kv_{Ti}] \sim \exp(-T_e/T_i) \ll 1$; ion Landau damping is negligible.

We next calculate the ion-acoustic dispersion relation, $\omega_j = \omega_{R,j} + i\gamma_j$, from the roots of the dielectric function of equation 4.49. We again apply the assumption that $\omega/kv_{Te} \ll 1$, and take $\omega \approx \omega_R$ in the imaginary term (the Landau damping term) of equation 4.49. That is, we assume that $\gamma_j \ll \omega_{R,j}$. Thus, to find the dispersion relation, we solve the equation

$$1 + \frac{1}{k^2\lambda_{De}^2} - \frac{\omega_{pi}^2}{(\omega_j - \mathbf{k}\cdot\mathbf{V}_i)^2} + i\Delta = 0, \tag{4.50}$$

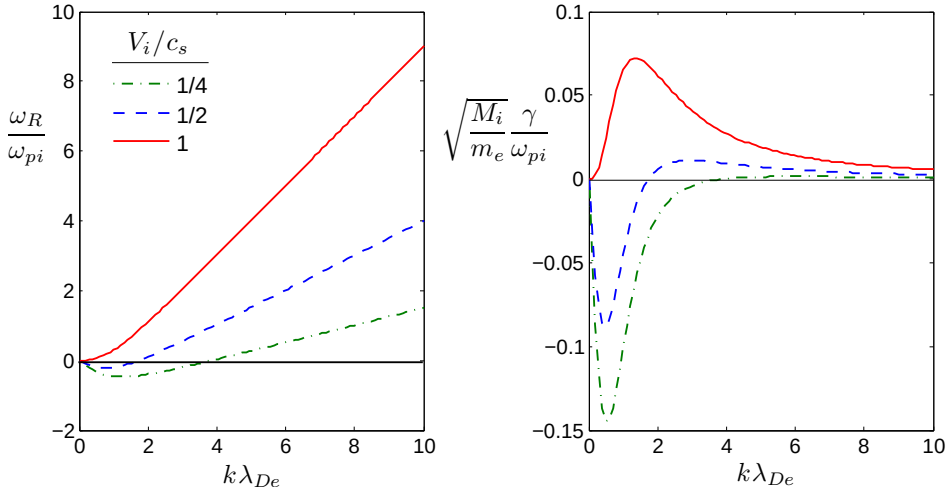

Figure 4.5: Plots of the real and imaginary parts of the dispersion relation, equation 4.52, for three values of the fluid speed (normalized to the sound speed) that are found in presheaths: 1/4 (green, dash-dotted line), 1/2 (blue, dashed line) and 1 (red, solid line). Positive γ represent growth rates. For the plots, we have used the approximation $\mathbf{k}\cdot\mathbf{V}_i \approx kV_i$.

in which $\Delta \equiv \sqrt{\pi}\omega_{R,j}/(k^3\lambda_{De}^2 v_{Te}) \ll 1$. Solving for $\omega_j - \mathbf{k}\cdot\mathbf{V}_i$ and expanding the result for $\Delta \ll 1$ yields

$$\omega_j - \mathbf{k}\cdot\mathbf{V}_i = \pm\frac{kc_s}{\sqrt{1+k^2\lambda_{De}^2(1+i\Delta)}} \approx \pm\frac{kc_s}{\sqrt{1+k^2\lambda_{De}^2}}\left(1 - i\frac{1}{2}\frac{k^2\lambda_{De}^2}{1+k^2\lambda_{De}^2}\Delta\right). \tag{4.51}$$

Inserting the definition of Δ, yields the ion-acoustic dispersion relation

$$\omega_\pm = \left(\mathbf{k}\cdot\mathbf{V}_i \pm \frac{kc_s}{\sqrt{1+k^2\lambda_{De}^2}}\right)\left(1 \mp i\frac{\sqrt{\pi m_e/8M_i}}{(1+k^2\lambda_{De}^2)^{3/2}}\right), \tag{4.52}$$

which consists of one growing and one damped ion-acoustic mode. A growing solution thus exists as long as the ion fluid speed is large enough: $|\mathbf{k}\cdot\mathbf{V}_i| > kc_s/\sqrt{1+k^2\lambda_{De}^2}$. Equation 4.52 is plotted in figure 4.5 for three representative values of the ion fluid speed in the presheath. Figure 4.5 shows that the relevant wavelength for unstable modes are near the electron Debye length (or shorter).

We made a couple of assumptions at the outset of deriving equation 4.52, and we now confirm that they are not contradicted by our final ion acoustic dispersion relation. One of these assumptions was that $\omega_j \sim kc_s$, which is easily confirmed from equation 4.52. The second assumption we made was that $\gamma_j \ll \omega_{R,j}$. Indeed, equation 4.52 confirms that

$$\frac{\gamma_j}{\omega_{R,j}} = \frac{\omega_{R,j}}{\omega_{R,j}}\frac{\sqrt{\pi m_e/8M_i}}{(1+k^2\lambda_{De}^2)^{3/2}} \sim \sqrt{\frac{m_e}{M_i}} \approx 1\times 10^{-3} \tag{4.53}$$

in which the last number assumes mercury ions. Thus, our original assumptions are consistent with the final dispersion relation.

4.4 Electron-Electron Scattering Lengths in the Presheath

Finally, we have the background and tools necessary to evaluate the electron-electron scattering frequency, and hence the scattering length, in the plasma-boundary transition region. In chapter 2 we found that the evolution of the distribution function for any species s is governed by the plasma kinetic equation $df_s/dt = \sum_{s'} C(f_s, f_{s'})$, in which $d/dt = \partial/\partial t + \mathbf{v}\cdot\partial/\partial\mathbf{x} + \mathbf{E}\cdot\partial/\partial\mathbf{v}$ is the convective derivative. Recall that $C(f_s, f_{s'})$ is the component collision operator describing the evolution of f_s due to collisions with each plasma species s' including itself ($s = s'$). The collision frequency thus scales with the magnitude of the collision operator $\nu^{s/s'} \sim C(f_s, f_{s'})/f_s$.

Here we are interested in determining the electron-electron collision frequency $\nu^{e/e}$ from both conventional Coulomb collisions and instability-enhanced collisions. Here the instabilities are the ion acoustic instabilities from equation 4.52. We showed in section 3.4.7 that the unique equilibrium distribution function is Maxwellian for these individual species collisions. We also showed that the approach to this unique equilibrium is determined by the timescale of the dominant contribution to the collision operator. This is either from conventional Coulomb collisions, which $C_{\mathrm{LB}}(f_e, f_e)$ describes, or from instability-enhanced collisions, which $C_{\mathrm{IE}}(f_e, f_e)$ describes. The instability-enhanced collisions were shown to drive the electron distribution to a unique Maxwellian as long as $\gamma_j^2/\omega_{R,j}^2 \ll 1$. From 4.53, we see that $\gamma_j^2/\omega_{R,j}^2 \sim 10^{-6}$ for ion acoustic instabilities. Thus, to high degree of accuracy, both conventional Coulomb collisions and instability-enhanced collisions drive the plasma to Maxwellian. If either term predicts an electron-electron scattering length shorter than the discharge length, we expect the electron distribution function to be Maxwellian.

Recall from equation 2.45 that the electron-electron collision operator is

$$C(f_e, f_e) = -\frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v' \, \mathcal{Q}^{e/e} \cdot \frac{1}{m_e}\left(f_e(\mathbf{v}) \frac{\partial f_e(\mathbf{v}')}{\partial \mathbf{v}'} - f_e(\mathbf{v}') \frac{\partial f_e(\mathbf{v})}{\partial \mathbf{v}} \right) \tag{4.54}$$

in which $\mathcal{Q}^{e/e} = \mathcal{Q}^{e/e}_{\mathrm{LB}} + \mathcal{Q}^{e/e}_{\mathrm{IE}}$. Estimating $\partial/\partial\mathbf{v} \sim v_{Te}$, the electron-electron collision frequency scales

as

$$\nu^{e/e} \sim \frac{n_e}{m_e v_{Te}^2}(Q_{\mathrm{LB}}^{e/e} + Q_{\mathrm{IE}}^{e/e}) \tag{4.55}$$

in which the scalars $Q_{\mathrm{LB}}^{e/e}$ and $Q_{\mathrm{IE}}^{e/e}$ represent the dominant contributions of the dyads $\mathcal{Q}_{\mathrm{LB}}^{e/e}$ and $\mathcal{Q}_{\mathrm{IE}}^{e/e}$. The electron-electron collision length can then be determined from the shorter of $\lambda_{\mathrm{LB}}^{e/e} \approx v_{Te}/\nu_{\mathrm{LB}}^{e/e}$, or $\lambda_{\mathrm{IE}}^{e/e} \approx v_{Te}/\nu_{\mathrm{IE}}^{e/e}$.

Note that we have taken the characteristic electron speed to be the electron thermal speed, although we are concerned with electrons on the tail of the distribution that have speeds a few times faster than that (recall from equation 4.43 that $v_{\|c}/v_{Te} \approx \sqrt{11} \approx 3$). However, since both the Lenard-Balescu and instability-enhanced collision operators have the same scaling with v, the thermal speed estimate can be used to compare the relative contribution from each scattering mechanism regardless of the particular speed. Accounting for the precise v dependence adds significant complexity to the analysis. Here we are not interested in this level of detail. We focus on a new physics result where we show that instability-enhanced collisions can be orders of magnitude more frequent than conventional Coulomb collisions in the presheath.

4.4.1 Stable Plasma Contribution

Evaluating the electron-electron collision frequency in a stable plasma $\nu_{\mathrm{LB}}^{e/e} \sim n_e Q_{\mathrm{LB}}^{e/e}/(m_e v_{Te}^2)$ requires determining the Lenard-Balescu collisional kernel $\mathcal{Q}_{\mathrm{LB}}^{e/e}$. Recall from equation 2.43 that

$$\mathcal{Q}_{\mathrm{LB}} = \frac{2q_s^2 q_{s'}^2}{m_s} \int d^3k \frac{\mathbf{kk}}{k^4} \frac{\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]}{\left|\hat{\varepsilon}(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})\right|^2}. \tag{4.56}$$

We use equation 4.45 to determine $\hat{\varepsilon}(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})$. For the electrons of interest, the argument of Z' in the ion term is very large

$$\frac{\mathbf{k}\cdot\mathbf{v} - \mathbf{k}\cdot\mathbf{V}_i}{k v_{Ti}} \sim \frac{v_{Te}}{v_{Ti}} \sim \sqrt{\frac{T_e}{T_i}}\sqrt{\frac{M_i}{m_e}} \sim 6 \times 10^4 \gg 1, \tag{4.57}$$

so we apply the asymptotic expansion of the Z' function for the ion term. For the electron term, the argument is close to unity for thermal particles $\omega/k v_{Te} \sim v_{Te}/v_{Te} \sim 1$, but the bulk of the electron distribution is a bit slower, so we take the small argument expansion for the electrons (accounting for the full Z' electron term leads to small modifications of the Coulomb logarithm). With these limits

applied, we find that the dielectric function is approximately adiabatic

$$\hat{\varepsilon}(\mathbf{k}, \mathbf{k}\cdot\mathbf{v}) \approx 1 + \frac{1}{k^2\lambda_{De}^2} - \frac{\omega_{pi}^2}{k^2 v_{Te}^2} \approx 1 + \frac{1}{k^2\lambda_{De}^2}. \tag{4.58}$$

We showed in section 1.1.5 that if the plasma dielectric function is adiabatic (i.e., of the form of equation 4.58) that the Lenard-Balescu collisional kernel reduces to the Landau collisional kernel

$$\mathcal{Q}_{\mathrm{L}} = \frac{2\pi q_s^2 q_{s'}^2}{m_s} \ln\Lambda \frac{u^2\mathcal{I} - \mathbf{uu}}{u^3}. \tag{4.59}$$

Since $\mathbf{u} \sim v_{Te}$ for electrons, the dominant contribution is

$$Q_{\mathrm{LB}}^{e/e} \approx \frac{2\pi e^4}{m_e v_{Te}} \ln\Lambda. \tag{4.60}$$

Putting this into the collision frequency estimate yields

$$\boxed{\nu_{LB}^{e/e} \approx \frac{\omega_{pe}}{8\pi n \lambda_{De}^3} \ln\Lambda.} \tag{4.61}$$

Recall that the relevant parameters of Langmuir's plasma were: $T_e = 2$ eV and $n_e = 10^{11}$ cm^{-3}. These parameters imply that the electron plasma frequency was $\omega_{pe} = 1.8 \times 10^{10}$ s^{-1} and the electron Debye length was $\lambda_{De} = 3 \times 10^{-5}$ m. The number of electrons in a Debye cube was $n_e\lambda_{De}^3 = 2700$ and the Coulomb logarithm was $\ln\Lambda \approx \ln(12\pi n_e \lambda_{De}^3) = 11.5$. Applying these numbers to equation 4.61, we find that $\nu_{\mathrm{LB}}^{e/e} \sim 3 \times 10^6$ s^{-1} and

$$\boxed{\lambda_{\mathrm{LB}}^{e/e} \approx \frac{v_{Te}}{\nu_{\mathrm{LB}}^{e/e}} \approx 28 \text{ cm}.} \tag{4.62}$$

The value that Langmuir predicted for this plasma in 1925 was 30 cm [13]. Although the kinetic theory of stable plasmas progressed significantly after Langmuir, the estimated electron-electron scattering length from the refined theories is not significantly shorter than Langmuir's estimate (actually the two estimates are so close that it is almost a coincidence – we have used rather crude estimates in obtaining equation 4.62 and even if this prediction were different from Langmuir's by a factor of two, or more, we would consider them to essentially agree). Stable plasma kinetic theory cannot explain why the electron-electron scattering length was less than 3 cm in Langmuir's discharge.

4.4.2 Ion-Acoustic Instability-Enhanced Contribution

Next, we consider the instability-enhanced contribution to the electron-electron collision frequency $\nu_{\mathrm{IE}}^{e/e} \sim n_e Q_{\mathrm{LB}}^{e/e}/(m_e v_{Te}^2)$. Recall from equation 2.47 that for slowly growing instabilities, which satisfy

$\gamma_j/\omega_{R,j} \ll 1$, the instability-enhanced collisional kernel is given by

$$\mathcal{Q}_{\mathrm{IE}} \approx \sum_j \frac{2q_s^2 q_{s'}^2}{m_s} \int d^3k \frac{\mathbf{kk}}{k^4} \frac{\pi\, \delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]\, \delta(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v}) \exp(2\gamma_j t)}{\gamma_j \left|\partial\hat{\varepsilon}(\mathbf{k},\omega)/\partial\omega\right|^2_{\omega_j}}. \tag{4.63}$$

Equation 4.53 shows that the ion-acoustic instabilities considered here are slowly growing, so that this approximation is valid.

From equation 4.49, we find

$$\frac{\partial\hat{\varepsilon}}{\partial\omega} = \frac{2\omega_{pi}^2}{(\omega - \mathbf{k}\cdot\mathbf{V}_i)^3} + i\frac{\sqrt{\pi}}{kv_{Te}} \frac{1}{k^2\lambda_{De}^2} \approx \frac{2\omega_{pi}^2}{(\omega - \mathbf{k}\cdot\mathbf{V}_i)^3}, \tag{4.64}$$

where the last step follows from the fact that the ω of interest satisfy $\omega \approx kc_s$. From the dispersion relation of equation 4.52, $\omega_j \approx \mathbf{k}\cdot\mathbf{V}_i - kc_s/\sqrt{1+k^2\lambda_{De}^2}$, and we find

$$\left|\frac{\partial\hat{\varepsilon}}{\partial\omega}\right|^2_{\omega_j} = \frac{4\omega_{pi}^2}{k^6 c_s^6}(1+k^2\lambda_{De}^2)^3 = \frac{4}{k^2c_s^2}\frac{(1+k^2\lambda_{De}^2)^3}{k^4\lambda_{De}^4}. \tag{4.65}$$

The second delta function in equation 4.63 can be estimated from the more elementary form written as a peaked Lorentzian

$$\delta(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v}) \approx \frac{1}{\pi}\frac{\gamma_j}{(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v})^2 + \gamma_j^2} \approx \frac{1}{\pi}\frac{\gamma_j}{k^2c_s^2}. \tag{4.66}$$

Putting equations 4.65 and 4.66 into equation 4.63 yields

$$\mathcal{Q}_{IE}^{e/e} \approx \frac{1}{2}\frac{e^4}{m_e}\int d^3k \frac{\mathbf{kk}}{k^4}\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]\frac{k^4\lambda_{De}^4}{(1+k^2\lambda_{De}^2)^3}e^{2\gamma t}. \tag{4.67}$$

Next, we evaluate $2\gamma t$ for the convective ion-acoustic waves. As described in section 2.4, the $\exp(2\gamma t)$ term in equation 4.67 must be calculated in the rest frame of the unstable mode. Since the ion-acoustic instability is convective,

$$2\gamma t = 2\int_{\mathbf{x}_o(\mathbf{k})}^{\mathbf{x}} d\mathbf{x}' \cdot \frac{\mathbf{v}_g\gamma}{|\mathbf{v}_g|^2} \tag{4.68}$$

in which $\mathbf{v}_g \equiv \partial\omega_R/\partial\mathbf{k}$ is the group velocity, $\mathbf{x}_o(\mathbf{k})$ is the location in space where wavevector $\mathbf{k}$ becomes unstable, and the integral $d\mathbf{x}'$ is taken along the path of the mode. An important consequence is that, since ω_- and $\mathbf{x}_o$ have no explicit time dependence, f_e will change with position, but not in time, in the laboratory frame. The plasma can thus remain in a steady-state and the EVDF will equilibrate to Maxwellian at a distance from the sheath determined by $\lambda^{e/e}(\mathbf{x})$.

In principle, the spatial integral in equation 4.68 requires integrating the profile of γ and $\mathbf{v}_g$, which change through the presheath due to variations in the ion fluid speed and the electron density, as

well as knowing the spatial location $\mathbf{x}_o(\mathbf{k})$ at which each wavevector $\mathbf{k}$ becomes excited. In estimating equation 4.68 we assume that changes due to spatial variations is weak, and we account for $\mathbf{x}_o(\mathbf{k})$ by only integrating over the unstable $\mathbf{k}$ for each spatial location $\mathbf{x}$. Following these approximations we obtain

$$2\gamma t \approx \frac{2Z\gamma}{v_g}, \tag{4.69}$$

in which Z is a shifted coordinate (with respect to z) that takes as its origin the location where the first instability onset occurs. In this case, $Z = 0$ will be the presheath-plasma boundary at $z = -2l$. Thus, we will have $Z = 2l + z$ (recall that by our convention z ranges from $-2l$ to 0 in the presheath).

The group speed of the ion acoustic waves is determined from equation 4.52 to be

$$v_g = V_i - \frac{c_s}{(1 + k^2\lambda_{De}^2)^{3/2}}. \tag{4.70}$$

Putting this into our approximation for $2\gamma t$ yields

$$2\gamma t \approx \frac{2Z\gamma}{v_g} = \frac{2kZ\sqrt{\pi m_e/8M_i}}{(1 + k^2\lambda_{De}^2)^{3/2}} \frac{V_i - c_s(1 + k^2\lambda_{De}^2)}{V_i - c_s} \approx \sqrt{\frac{\pi m_e}{2M_i}} \frac{Z}{\lambda_{De}} \frac{k\lambda_{De}}{(1 + k^2\lambda_{De}^2)^{3/2}}. \tag{4.71}$$

Returning to evaluating equation 4.67, we use spherical polar coordinates for $\mathbf{k}$, and take the parallel direction along $\mathbf{u}$, so that: $\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}') = k_\parallel(v - v') \approx k_\parallel v_{Te}$. Since the integrand does not depend on the azimuthal angle (from our approximations), the $k_\parallel$ and k_ϕ integrals are trivial to evaluate. After the $\delta(k_\parallel)$ integral, the k^2 terms are $k^2 = k_\perp^2$. We also apply the variable substitution $\kappa = k_\perp \lambda_{De}$. After these evaluations, equation 4.67 becomes

$$Q_{\mathrm{IE}}^{e/e} \approx \frac{\pi e^4}{m_e v_{Te}} \int_{\kappa_c}^{\infty} d\kappa \frac{\kappa^3}{(1 + \kappa^2)^3} \exp\left[\frac{Z}{\lambda_{De}} \sqrt{\frac{\pi m_e}{2M_i}} \frac{\kappa}{(1 + \kappa^2)^{3/2}}\right], \tag{4.72}$$

in which we have set the lower limit of integration to κ_c so that only the unstable k are integrated over. The limit κ_c can be determined from the instability criterion

$$V_i - \frac{c_s}{\sqrt{1 + \kappa^2}} > 0. \tag{4.73}$$

Setting this expression equal to zero yields

$$\kappa_c \equiv \begin{cases} \sqrt{c_s^2/V_i^2 - 1}\ , & \text{for } V_i \le c_s \\ 0\ , & \text{for } V_i \ge c_s \end{cases}. \tag{4.74}$$

The κ integral in equation 4.72 is very difficult to evaluate analytically. However, a good approximation can be obtained as follows. The integrand is peaked about the point where $\kappa^3/(1+\kappa^2)^3$ is a maximum, which is at $\kappa = 1$. Expanding the argument of the exponential about this point yields

$$\left.\frac{\kappa}{(1+\kappa^2)^{3/2}}\right|_{\kappa=1} = \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{8}(\kappa-1) - \frac{3\sqrt{2}}{32}(\kappa-1)^2 + \ldots \tag{4.75}$$

Keeping only the lowest order term, we use the approximation $\kappa/(1+\kappa^2)^{3/2} \approx \sqrt{2}/4$ in the exponential. The integrand is then algebraic, and can be evaluated analytically

$$\int_{\kappa_c}^{\infty} d\kappa \frac{\kappa^3}{(1+\kappa^2)^3} = \frac{1}{4}\frac{1+2\kappa_c^2}{(1+\kappa_c^2)^2}. \tag{4.76}$$

Putting this integral approximation into equation 4.72, we find

$$Q_{\mathrm{IE}}^{e/e} \approx \frac{\pi e^4}{4 m_e v_{Te}} \frac{1+2\kappa_c^2}{(1+\kappa_c^2)^2} \exp\left(\eta \frac{Z}{l}\right), \tag{4.77}$$

in which l is a length scale characterizing the presheath; typically it is the ion-neutral collision mean free path. We have also defined the dimensionless coefficient

$$\eta \equiv \sqrt{\frac{\pi m_e}{8 M_i}} \frac{l}{\lambda_{De}}. \tag{4.78}$$

The instability-enhanced collisional kernel can also be expressed in terms of a multiple of the Lenard-Balescu collisional kernel

$$Q_{\mathrm{IE}}^{e/e} \approx \frac{Q_{\mathrm{LB}}^{e/e}}{8 \ln\Lambda} \frac{1+2\kappa_c^2}{(1+\kappa_c^2)^2} \exp\left(\eta \frac{Z}{l}\right). \tag{4.79}$$

Thus, the effective collision frequency due to instability-enhanced collective interactions is given by

$$\boxed{\nu_{\mathrm{IE}}^{e/e} \approx \frac{\nu_{\mathrm{LB}}^{e/e}}{8 \ln\Lambda} \frac{1+2\kappa_c^2}{(1+\kappa_c^2)^2} \exp\left(\eta \frac{Z}{l}\right).} \tag{4.80}$$

The corresponding electron-electron collision length is $\lambda_{\mathrm{IE}}^{e/e} \approx v_{Te}/\nu_{\mathrm{IE}}^{e/e}$. The location $Z = 0$ corresponds to the spatial location where instability onset occurs (at the presheath-plasma boundary here). The presheath solution is typically valid over the domain: $z : -2l \to 0$ if l is taken as the ion-neutral collision length [84]. For $z \lesssim -2l$, the presheath electric field is essentially zero and the ion fluid flow is also zero: this is the bulk plasma. Thus, since we take z to have negative values in the presheath, the spatial variable with its origin at the onset location of instabilities is related to z by: $Z = 2l + z$.

For Langmuir's discharge parameters $l \approx 11$ cm [84], and $\eta \approx 4.5$. In figure 4.6, we plot the total predicted electron-electron scattering frequency, along with the individual contributions from stable

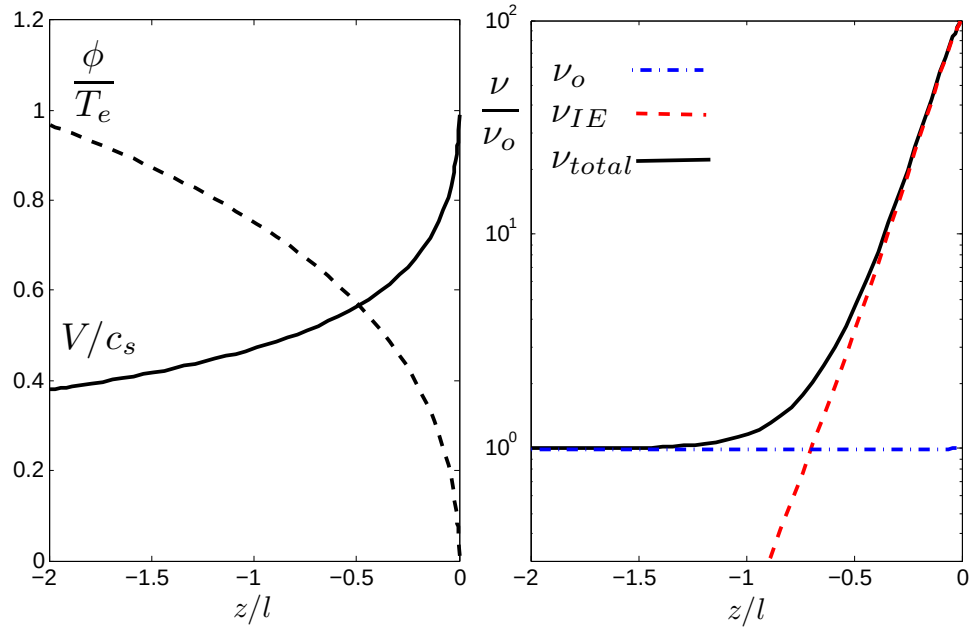

Figure 4.6: Left: the electrostatic potential drop in the presheath normalized to the electron temperature, ϕ/T_e, (dashed line) and corresponding ion flow speed normalized to the ion sound speed, V_i/c_s, (solid line). Right: the total electron-electron scattering frequency normalized to the stable plasma collision frequency (solid line). Also shown are the individual contributions from Coulomb interactions in a stable plasma (blue, dashed-dotted line) and from instability-enhanced collective interactions (red dashed line). Here we have used the constant ion-neutral collision frequency model, $\nu^{i/n}$ =constant, for the presheath; see equations 4.33 and 4.34.

plasma theory and the instability enhancements. Here we have used the constant ion-neutral collision frequency model, $\nu^{i/n}$ =constant, for the presheath from equations 4.33 and 4.34. However, the results are not sensitive to which presheath model is used. Figure 4.6 shows that near the sheath-presheath boundary ion-acoustic instabilities enhance the electron-electron scattering approximately 100 times the nominal stable plasma rate. The collision length for electron-electron scattering is shortened by more than a factor of 10 over a distance of approximately $l/2$. Thus, near the plasma boundary, instability-enhanced collective interactions determine the scattering rate and drive the plasma toward the unique Maxwellian EVDF within a presheath-scale distance. Thus, instability-enhance scattering, due to ion acoustic instabilities in the presheath, can explain Langmuir's measurements.

Finally, we consider the validity of the assumption of linearly growing waves determined by equation 2.130 and also when the instability-enhancement dominates the conventional Coulomb interactions. Putting in the dielectric function and ion-acoustic instability dispersion relation into equation 2.130, and using $z = l$, we find that the theory is valid as long as

$$\frac{1}{n\lambda_{De}^3}\sqrt{\frac{4\pi}{\eta z/l}}\exp\left(\frac{\eta}{\sqrt{2}}\frac{Z}{l}\right) \lesssim 1. \tag{4.81}$$

In which we have estimated the integral in equation 4.72 assuming the argument of the exponential is large. Comparing $\mathcal{Q}_{\mathrm{LB}}^{e/e}$ and $\mathcal{Q}_{\mathrm{IE}}^{e/e}$, we find that instability-enhanced collisions dominate when

$$\frac{1}{8\ln\Lambda}\sqrt{\frac{\pi/2}{\eta z/l}}\exp\left(\frac{\eta}{2\sqrt{2}}\frac{Z}{l}\right) \gtrsim 1. \tag{4.82}$$

Equations 4.81 and 4.82 determine a region depending on plasma parameter and η for which instability-enhanced collisions for instabilities in a linear growth regime are both valid and dominant. This is shown as the shaded region in figure 4.7. The blue line in figure 4.7 represents the parameters in the presheath of Langmuir's discharge. For this plasma with $n\lambda_{De}^3 \approx 3{\times}10^3$, the theory is valid for $\eta z/l \lesssim 55$. In this presheath example the maximum $\eta z/l \lesssim 10$, which is reached at the sheath-presheath boundary; thus, the theory is well-suited to this problem.

Aspects of the model proposed in this chapter can be directly tested experimentally. The k-space fluctuations could be characterized in the presheath. We predict that modes satisfying $k \gtrsim 1/\lambda_{De}$ become unstable and grow exponentially toward the boundary. These fluctuations should disappear due to ion Landau damping if the ions are heated to $T_i \approx T_e$. Alternatively, accounting for ion-neutral

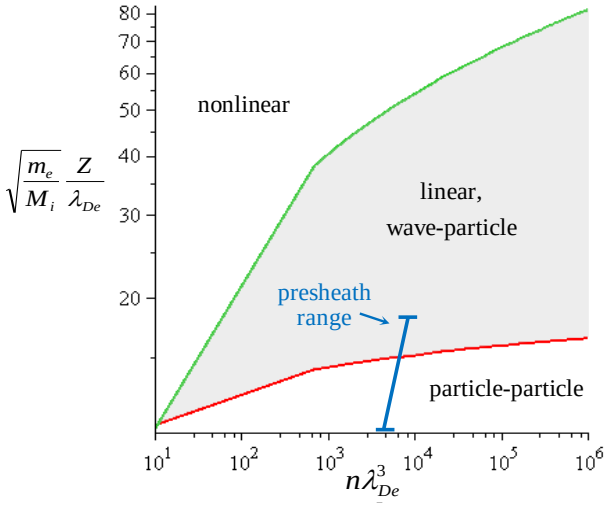

Figure 4.7: Plot of equations 4.81 and 4.82 which determine the shaded region in which the plasma kinetic theory is valid and where instability-enhanced interactions dominate. The theory is valid for all values below the green line. Instabilities dominate scattering for all values above the red line.

damping results in a $-i\nu^{i/n}/2$ term to be added to equation 4.52. Using $\nu^{i/n} \approx c_s/\lambda^{i/n}$, leads to the result that the ion-acoustic instabilities are ion-neutral damped for $\eta \lesssim 1$. Since $\eta > 1$ is required for instability enhanced scattering, this also represents a maximum neutral density above which the presheath length is so short that the instabilities have an insufficient distance to grow before reaching the boundary. Experimentally, electron scattering could thus be attributed to instability-enhanced collective interactions by measuring both the fluctuations and the EVDF with and without instabilities.

Chapter 5

Kinetic Theory of the Presheath and the Bohm Criterion

In section 4.2.2, the Bohm criterion was derived. This derivation (which was originally provided by Bohm in reference [38]) assumes that all ions have the same velocity, denoted by $\mathbf{V}$, in the direction perpendicular to the boundary surface. The resulting criterion requires that this speed satisfy $V \geq c_s$ at the sheath edge, here $c_s \equiv \sqrt{T_e/M_i}$ is the ion sound speed. This approach effectively assumes a delta-function distribution for ions $f_i = n_i \delta(\mathbf{v} - \mathbf{V})$ and neglects any thermal motion. It also assumes that electrons obey the Boltzmann relation. An important question to answer is; how does the conventional Bohm criterion change when more general electron and ion distribution functions are taken into account?

Attempts to answer this question have been the topic of several papers over the past 50 years [42, 86–94]. However, essentially no experiments have been performed to test the theories that have been proposed. This is an unfortunate situation because the theoretical proposal that has come to prominence does not give a meaningful criterion for many common distribution functions. The result that is often quoted (see for example [42, 94]) is

$$\frac{1}{M_i} \int d^3v \frac{f_i(\mathbf{v})}{v_z^2} \leq -\frac{1}{m_e} \int d^3v \, \frac{1}{v_z} \frac{\partial f_e(\mathbf{v})}{\partial v_z}, \tag{5.1}$$

and is commonly called the "generalized Bohm criterion." It is even cited prominently in a popular plasma physics textbook [80]. Here, the electric field of the sheath is taken to be aligned in the $\hat{z}$ direction, and it is assumed that the only spatial gradients of f are caused by this electric field.

Although it is frequently cited, equation 5.1 does not produce a meaningful criterion for most plasmas of interest. If the ion distribution function has any particles with zero velocity, the left side of equation 5.1 diverges. If the velocity gradient of the electron distribution does not vanish for $v_z = 0$, the right

side of equation 5.1 diverges. For example, if the ion distribution function is Maxwellian, the left side of equation 5.1 is ∞. Similarly, the right side can diverge for certain distribution functions, even when the left side is finite: examples of this are discussed in section 5.4. Equation 5.1 places unphysical importance on the part of the distribution functions where particles are slow. Despite the fact that it often gives unphysical results, and that this shortcoming has been pointed out before [80, 87], equation 5.1 continues to be used in plasma physics literature [93, 94].

In section 5.1, we reconsider previous derivations of the generalized Bohm criterion given by equation 5.1. We show that these derivations contain two errors. The first of these is taking the v_z^{-1} moment of the collisionless kinetic equation (i.e. Vlasov equation). Neglecting the v_z^{-1} moment of the collision operator is a mistake because it diverges when the distribution functions have particles near zero velocity (just like the term on the left side of equation 5.1). Only velocity moments with a positive power can be applied to the Vlasov equation, or divergences will result for $v_z = 0$. The second error is a mathematical mistake where integration by parts is misapplied to a function that is not continuously differentiable. This error can easily be corrected, but the resultant criterion then differs from equation 5.1.

In section 5.2, we derive an alternative form of a generalized Bohm criterion that is based upon moments of the kinetic equation in which the velocity multiplier has only positive powers (rather than the v_z^{-1} moment of previous work). This approach avoids the possibility of diverging results. Our result supports previous derivations of the Bohm criterion based on fluid theory, and it returns these results in the fluid limit. Particles with low energy do not have any special significance in our theory. In contrast, equation 5.1 does not return the fluid results in the appropriate limit because it places undue importance on low energy particles. In section 5.3, we comment on ion-ion collisions in the presheath; specifically, on how ion-acoustic instabilities can play a significant role in determining the ion distribution function. This effect is similar to how ion-acoustic instabilities cause the electron distribution function to become Maxwellian in the presheath; as was discussed in chapter 4. Finally, in section 5.4, we consider a couple of example distribution functions that are common in low temperature plasmas, but for which the generalized Bohm criterion we derive in section 5.2 gives significantly different predictions than equation 5.1.

5.1 Previous Kinetic Theories of the Bohm Criterion

The first variant of equation 5.1 that appeared in the literature was a 1959 paper by Harrison and Thompson [86]. Equation (21) of that work gave the result

$$M_i\left(\int d^3v \frac{1}{v_z^2}\frac{f_i(\mathbf{v})}{n_i}\right)^{-1} \geq T_e. \tag{5.2}$$

Equation 5.2 is the same as equation 5.1 if one assumes a stationary Maxwellian distribution for electrons. Shortly after Harrison and Thompson's publication, Hall pointed out it's deficiencies [87], specifically citing that it "ascribes undue importance to the presence of low velocity ions at the sheath edge." Despite this criticism, Harrison and Thompson's work quickly caught on and it has become widely used [88–94]. More recently, the electron term was also generalized [42] to give the modern form of equation 5.1 and the result has come to greater prominence through its mention in a popular review article by Riemann [39].

5.1.1 The Sheath Condition

In order to derive equation 5.1, one must first develop a mathematical definition of the sheath edge (the interface between the sheath and the quasineutral plasma, or presheath) [39]. This is typically called the "sheath criterion." The sheath criterion is based on the physical condition that as one moves from the sheath to the plasma, the plasma becomes quasineutral and this location is defined as the sheath edge. Expanding Gauss's law about about the sheath edge, near $\phi = 0$, yields

$$\frac{d^2\phi}{dz^2} = -4\pi\left[\underbrace{\rho(\phi=0)}_{=0} + \left.\frac{d\rho}{d\phi}\right|_{\phi=0}\phi + \ldots\right]. \tag{5.3}$$

Recall that $\rho \equiv \sum_s q_s n_s$. At the marginal condition where quasineutrality is met, the $d\rho/d\phi$ term of equation 5.3 dominates. Multiplying equation 5.3 by $d\phi/dz$ and integrating with respect to ϕ gives the relation

$$\frac{E^2}{4\pi} + \left.\frac{d\rho}{d\phi}\right|_{\phi=0}\phi^2 = C \tag{5.4}$$

in which C is a constant. Since $\phi \to 0$ as $z/\lambda_{De} \to \infty$ on the sheath length scale, the constant C must be zero. We are then left with

$$\left.\frac{d\rho}{d\phi}\right|_{\phi=0} = -\frac{E^2}{4\pi\phi^2} \tag{5.5}$$

which implies the sheath criterion

$$\boxed{\left.\frac{d\rho}{d\phi}\right|_{\phi=0} \leq 0.} \tag{5.6}$$

The sheath criterion is a succinct mathematical definition of the sheath edge [39]. Using the fact that

$$\frac{dn_s}{d\phi} = \frac{dn_s}{dz}\frac{dz}{d\phi} = -\frac{1}{E}\frac{dn_s}{dz}, \tag{5.7}$$

the sheath condition can also be written

$$\sum_s q_s \left.\frac{dn_s}{dz}\right|_{z=0} \geq 0. \tag{5.8}$$

Since the relation between density and the distribution function is simply $n_s = \int d^3v\, f_s$, this criterion can be given in terms of the distribution function

$$\sum_s q_s \int_{-\infty}^{\infty} d^3v\, \frac{\partial f_s}{\partial z} \geq 0. \tag{5.9}$$

5.1.2 Previous Forms of Kinetic Bohm Criteria

Previous kinetic theories of the Bohm criterion are collisionless, being based on the Vlasov equation

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\frac{\partial f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s}\mathbf{E}\cdot\frac{\partial f_s}{\partial \mathbf{v}} = 0. \tag{5.10}$$

Since we are considering steady state, the $\partial/\partial t$ term can be set to zero. We take $\mathbf{E} = E\hat{z}$ and assume that the only spatial gradients of f_s are due to this electric field, so they are in the $\hat{z}$ direction as well. This leaves the 1-D, steady-state version of the Vlasov equation

$$v_z\frac{\partial f_s}{\partial z} + \frac{q_s}{m_s}E\frac{\partial f_s}{\partial v_z} = 0. \tag{5.11}$$

The next step that is taken in previous kinetic formulations is to divide equation 5.11 by v_z, to obtain an expression for $\partial f_s/\partial z$, then insert the result into the sheath criterion of equation 5.9. In other words, they take the v_z^{-1} moment of the Vlasov equation. Doing so yields the condition

$$\sum_s \frac{q_s^2}{m_s}\int_{-\infty}^{\infty} d^3v\, \frac{1}{v_z}\frac{\partial f_s}{\partial v_z} \leq 0, \tag{5.12}$$

which is a form of a generalized Bohm criterion. Assuming that the plasma consists of a single species of ions with unit charge and electrons, this is

$$\frac{1}{M_i}\int_{-\infty}^{\infty} d^3v \frac{1}{v_z}\frac{\partial f_i}{\partial v_z} \leq -\frac{1}{m_e}\int_{-\infty}^{\infty} d^3v \frac{1}{v_z}\frac{\partial f_e}{\partial v_z}. \tag{5.13}$$

One final step is conventionally performed in order to write this in the form of equation 5.1, and that is to integrate the ion term by parts,

$$\int_{-\infty}^{\infty} dv_z \frac{1}{v_z}\frac{\partial f_s}{\partial v_z} = \underbrace{\int_{-\infty}^{\infty} dv_z \frac{\partial}{\partial v_z}\left(\frac{1}{v_z} f_i\right)}_{=0} + \int_{-\infty}^{\infty} dv_z \frac{1}{v_z^2} f_i. \tag{5.14}$$

Taking the surface term to be zero, we are left with the conventional form of the generalized Bohm criterion

$$\frac{1}{M_i}\int d^3v \frac{f_i(\mathbf{v})}{v_z^2} f_i \leq -\frac{1}{m_e}\int d^3v \frac{1}{v_z}\frac{\partial f_e(\mathbf{v})}{\partial v_z}, \tag{5.15}$$

which is the same as was quoted in equation 5.1. If the electrons are taken to be a stationary Maxwellian distribution, equation 5.15 reduces to Harrison and Thompson's equation 5.2

$$M_i\left(\int d^3v \frac{1}{v_z^2}\frac{f_i(\mathbf{v})}{n_i}\right)^{-1} \geq T_e. \tag{5.16}$$

5.1.3 Deficiencies of Previous Kinetic Bohm Criteria

Two mistakes are made in the previous derivations of a kinetic Bohm criterion which lead to the unphysical divergences in equations 5.15 and 5.16 that occur when the distribution functions have any contribution at zero velocity. These same mistakes are also present in the summarized version of these derivations that was presented in the last section. They are

(1) The collision operator should not be neglected if one is to take the v_z^{-1} moment of the kinetic equation. This is because $\int d^3v\, C(f_s)/v_z$ diverges unless the collision operator is zero. The collision operator is only zero if the plasma is in equilibrium. However, if the plasma is in equilibrium electrons and ion must both have Maxwellian distribution functions with equal temperatures and flow speeds (recall section 3.4.7), but such a distribution function cannot be a solution near the sheath edge because of the presence of the presheath electric field.

(2) Since the function $(1/v_z)\partial f_s/\partial v_z$ is typically not continuously differentiable, the integration by parts conducted in equation 5.14 in invalid.

The easier of these two issues to correct is (2), since the integration by parts step shown in equation 5.14 can simply be avoided and the generalized Bohm criterion left in the form of equation 5.13. However, even this equation is incorrect because of issue (1), as we will discuss next. That the integration by

parts step of equation 5.14 is incorrect, can by shown using a simple example. The contentious step is of the form

$$\int_{-\infty}^{\infty} dx \frac{1}{x}\frac{df}{dx} = \underbrace{\int_{-\infty}^{\infty} dx \frac{d}{dx}\left(\frac{1}{x}f\right)}_{=0} + \int_{-\infty}^{\infty} dx \frac{1}{x^2} f, \tag{5.17}$$

for any physically possible distribution function f (e.g., the restrictions $f(\pm\infty) = 0$ and that f is always positive can be imposed since any meaningful plasma distribution must obey these). If one takes as an example, $f = \exp(-x^2)$, the left side of equation 5.17 can be evaluated directly

$$\int_{-\infty}^{\infty} dx \frac{1}{x}\frac{df}{dx} = -2\int_{-\infty}^{\infty} dx\, e^{-x^2} = -2\sqrt{\pi}. \tag{5.18}$$

However, if the surface term on the right side of equation 5.17 is taken to be zero, as is assumed in the previous theories, the right side of equation 5.17 diverges

$$\begin{aligned}\int_{-\infty}^{\infty} dx \frac{1}{x^2} e^{-x^2} &= \lim_{\epsilon\to 0}\left(\int_{-\infty}^{-|\epsilon|} dx \frac{1}{x^2} e^{-x^2} + \int_{|\epsilon|}^{\infty} dx \frac{1}{x^2} e^{-x^2}\right) \\ &= -2\sqrt{\pi} + \lim_{\epsilon\to 0}\left(\frac{2}{|\epsilon|} e^{-|\epsilon|^2} + 2\sqrt{\pi}\mathrm{erf}(|\epsilon|)\right) = -2\sqrt{\pi} + \lim_{\epsilon\to 0}\frac{2}{|\epsilon|} e^{-|\epsilon|^2} \to \infty.\end{aligned} \tag{5.19}$$

The reason that one cannot apply integration by parts to a function of the form of equation 5.17 is that integration by parts is only valid for continuously differentiable functions (see for example Rudin's book on analysis [95]). However, f'/x is not continuous unless $f'(x = 0) = 0$, and f'/x is not continuously differentiable unless both f''/x and f'/x^2 are continuous. Thus, issue (2) restricts the previous kinetic Bohm criteria to the form of equation 5.13. However, issue (1) shows that there are problems with equation 5.13 as well.

Equation 5.13 still contains divergences that lead to meaningless criteria at the sheath edge. For example, if the ion distribution function is taken to be a flow-shifted Maxwellian with flow speed $\mathbf{V} = V\hat{z}$ and the electron distribution function is taken to be a stationary Maxwellian, then equation 5.13 gives the criterion $-n_i/T_i + \infty \leq n_e/T_e$. This is obviously not a meaningful condition that ions must satisfy as they leave a plasma. The primary deficiency of the collisionless Vlasov approach used by previous authors, and outlined in section 5.1.2, is simply that the collision operator cannot be neglected if one is interested in v_z^{-1} moments of the kinetic equation.

Consider what happens if the collision operator is not neglected in the conventional derivation of the kinetic Bohm criterion. Then, the relevant kinetic equation has the form

$$v_z \frac{\partial f_s}{\partial z} + \frac{q_s}{m_s} E \frac{\partial f_s}{\partial v_z} = C(f_s). \tag{5.20}$$

Taking the v_z^{-1} moment of this, in order to find an equation for $\partial f_s/\partial z$, gives

$$\int_{-\infty}^{\infty} d^3v \, \frac{\partial f_s}{\partial z} = \int_{-\infty}^{\infty} d^3v \left[\frac{C(f_s)}{v_z} - \frac{q_s}{m_s} E \frac{1}{v_z} \frac{\partial f_s}{\partial v_z} \right]. \tag{5.21}$$

Putting this into the sheath criterion of equation 5.9 yields the condition

$$\sum_s \frac{q_s^2}{m_s} \int_{-\infty}^{\infty} d^3v \frac{1}{v_z} \frac{\partial f_s}{\partial v_z} \leq \sum_s \frac{q_s}{E} \int_{-\infty}^{\infty} d^3v \frac{1}{v_z} C(f_s), \tag{5.22}$$

which can be compared to the Vlasov result from equation 5.12. A brief study of equation 5.22 shows that not only the left side, but also the right side, which depends on the v_z^{-1} moment of the collision operator, diverges if $\partial f_s/\partial v_z \neq 0$ for $v_z = 0$ and any s. Equation 5.22 shows that neglecting the collision operator is not a consistent approximation when the v_z^{-1} is taken. For example, consider a plasma with a single stationary Maxwellian electron species and a single ion species with a flow relative to the electrons. In this case $C(f_i, f_i) = 0$ and $C(f_e, f_e) = 0$, but $C(f_e, f_i) \neq 0$. Since $f_i(v_z = 0) \neq 0$, the $C(f_e, f_i)$ term will cause the right side of equation 5.22 to diverge. The ion term on the left side of equation 5.22 diverges for this example as well.

This section has pointed out problems with previous kinetic Bohm criteria that are based on v_z^{-1} moments of the collisionless Vlasov equation. The result of this approach leads to sheath conditions that are impossible to use because individual terms can diverge if the distribution functions have particles with zero velocity. These divergences can be avoided if one builds a hierarchy of fluid moment equations from $\mathbf{v}^{|m|}$ moments of the kinetic equation. With this approach, the different moments such as fluid flow velocity, pressure and stress have unique definitions in terms of the distribution functions, and are also physically meaningful definitions of macroscopic quantities that characterize the plasma.

5.2 A Kinetic Bohm Criterion from Velocity Moments of the Kinetic Equation

In this section we derive a kinetic Bohm criterion from $v^{|m|}$ moments of the kinetic equation. By taking the moments $m = 1, 2, \ldots$, a complete set of fluid equations can be built from the kinetic equation. These moment equations are the same as the conventional plasma fluid equations, but the fluid parameters (such as flow velocity, temperature, pressure, etc.) are defined in terms of the moments of the distribution functions. In this way, the theory retains a kinetic interpretation. Fluid equations built from moments of the kinetic equation typically suffer from the drawback that the equation for each m formally depends on knowing the fluid variable associated with the $m+1$ moment. However, since this is a kinetic approach, a closure can be provided by simply writing a fluid variable (such as the stress tensor) in terms of its definition as a moment of f_s (instead of solving higher and higher order moments for new fluid variables). We will find that for most applications, a conventional fluid derivation of the Bohm criterion provides an excellent approximation to the kinetic result. Because the fluid variables are defined in terms of moments of f_s, the fluid result can also be written in terms of a condition on the distribution functions f_s.

5.2.1 Fluid Moments of the Kinetic Equation

In this chapter, we will be concerned with macroscopic (fluid) properties of the plasma in the plasma-boundary transition region, which are defined by velocity-space moments of the distribution functions. We used briefly in section 4.2 an approximate fluid model to describe the presheath. Here we develop a more complete fluid model that also accounts for collisional effects. To build this set of fluid equations, we start from the conservative form of the kinetic equation

$$\frac{\partial f_s}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{v} f_s + \frac{q_s}{m_s} \frac{\partial}{\partial \mathbf{v}} \cdot \Big(\mathbf{E} + \mathbf{v} \times \mathbf{B} \Big) f_s = C(f_s) \tag{5.23}$$

in which the collision operator $C(f_s) \equiv \sum_{s'} C(f_s, f_{s'})$ accounts for collisions with all species in the plasma $\sum_{s'}$ including itself ($s = s'$). We apply the following definitions for fluid variables of each species in terms of the distribution function for that species:

$$\text{Density:} \quad n_s(\mathbf{x}, t) \equiv \int d^3v \; f_s(\mathbf{x}, \mathbf{v}, t), \tag{5.24}$$

$$\text{Fluid flow velocity:} \quad \mathbf{V}_s(\mathbf{x},t) \equiv \frac{1}{n_s}\int d^3v\ \mathbf{v} f_s(\mathbf{x},t), \tag{5.25}$$

$$\text{Pressure tensor:} \quad \mathcal{P}_s(\mathbf{x},t) \equiv \int d^3v\ m_s \mathbf{v}_r\mathbf{v}_r f_s(\mathbf{x},t) = p_s\mathcal{I} + \Pi_s, \tag{5.26}$$

$$\text{Pressure (scalar):} \quad p_s(\mathbf{x},t) \equiv \int d^3v\ \frac{1}{3}m_s v_r^2 f_s(\mathbf{x},\mathbf{v},t) = n_s(\mathbf{x},t)T_s(\mathbf{x},t), \tag{5.27}$$

$$\text{Stress tensor:} \quad \Pi_s(\mathbf{x},t) \equiv \int d^3v\ m_s\left(\mathbf{v}_r\mathbf{v}_r - \frac{1}{3}v_r^2\mathcal{I}\right) f_s(\mathbf{x},t), \tag{5.28}$$

$$\text{Temperature:} \quad T_s(\mathbf{x},t) \equiv \frac{1}{n_s}\int d^3v\ \frac{1}{3}m_s v_r^2 f_s(\mathbf{x},t) = \frac{1}{2}m_s v_{Ts}^2, \tag{5.29}$$

$$\text{Conductive heat flux:} \quad \mathbf{q}_s(\mathbf{x},t) \equiv \int d^3v\ \frac{1}{2}m_s\mathbf{v}_r v_r^2 f_s(\mathbf{x},t), \tag{5.30}$$

$$\text{Frictional force density:} \quad \mathbf{R}_s(\mathbf{x},t) \equiv \int d^3v\ m_s\mathbf{v}C(f_s), \tag{5.31}$$

$$\text{Energy exchange density:} \quad Q_s(\mathbf{x},t) \equiv \int d^3v\ \frac{1}{2}m_s v_r^2 C(f_s), \tag{5.32}$$

in which we have defined a relative velocity $\mathbf{v}_r \equiv \mathbf{v} - \mathbf{V}_s$, where $\mathbf{V}_s$ is the fluid flow velocity from equation 5.25.

The density moment ($\int d^3v \ldots$) of the kinetic equation yields the continuity equation

$$\frac{\partial n_s}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\cdot\left(n_s\mathbf{V}_s\right) = 0. \tag{5.33}$$

The momentum moment ($\int d^3v\, m_s\mathbf{v} \ldots$) yields the momentum evolution equation

$$m_s n_s\left(\frac{\partial \mathbf{V}_s}{\partial t} + \mathbf{V}_s\cdot\frac{\partial \mathbf{V}_s}{\partial \mathbf{x}}\right) = n_s q_s\left(\mathbf{E} + \mathbf{V}_s\times\mathbf{B}\right) - \frac{\partial p_s}{\partial \mathbf{x}} - \frac{\partial}{\partial \mathbf{x}}\cdot\Pi_s + \mathbf{R}_s. \tag{5.34}$$

The energy moment ($\int d^3v\, m_s v^2 \ldots$) yields the energy evolution equation

$$\frac{\partial}{\partial t}\left(\frac{3}{2}n_s T_s + \frac{1}{2}m_s n_s V_s^2\right) + \frac{\partial}{\partial \mathbf{x}}\cdot\left[\mathbf{q}_s + \left(\frac{5}{2}n_s T_s + \frac{1}{2}n_s m_s V_s^2\right)\mathbf{V}_s + \mathbf{V}_s\cdot\Pi_s\right] - n_s q_s \mathbf{V}_s\cdot\mathbf{E} - Q_s - \mathbf{V}_s\cdot\mathbf{R}_s = 0. \tag{5.35}$$

Continuing this process with higher order moments of the kinetic equation leads to a hierarchy of fluid equations. Also, note that by applying the continuity and momentum evolution equations 5.33 and 5.34, the energy evolution equation can be written as a pressure evolution equation

$$\frac{3}{2}\frac{\partial p_s}{\partial t} = -\frac{\partial}{\partial \mathbf{x}}\cdot\left(\mathbf{q}_s + \frac{5}{2}p_s\mathbf{V}_s\right) + \mathbf{V}_s\cdot\frac{\partial p_s}{\partial \mathbf{x}} - \Pi_s : \frac{\partial}{\partial \mathbf{x}}\mathbf{V}_s + Q_s \tag{5.36}$$

or a temperature evolution equation

$$\frac{3}{2}n_s\frac{\partial T_s}{\partial t} = -n_s T_s\frac{\partial}{\partial \mathbf{x}}\cdot\mathbf{V}_s - \frac{3}{2}n_s\mathbf{V}_s\cdot\frac{\partial T_s}{\partial \mathbf{x}} - \frac{\partial}{\partial \mathbf{x}}\cdot\mathbf{q}_s - \Pi_s : \frac{\partial}{\partial \mathbf{x}}\mathbf{V}_s + Q_s. \tag{5.37}$$

Equations 5.33, 5.34, 5.35, and the subsequent equations built from higher-velocity moments of the kinetic equation, constitute a hierarchy of fluid equations. The utility of building the fluid equations this way is that we have defined the fluid variables in terms of the distribution function for each species. In this way, the hierarchy of fluid equations is as general as the kinetic equation itself. In the next section, we use equations 5.33 and 5.34 to formulate a Bohm criterion that is more general than the one originally proposed by Bohm (which assumed monoenergetic ions and Maxwellian electrons).

5.2.2 The Bohm Criterion

Before setting off on a calculation, we need to first establish what sort of expression we are trying to find. That is, what do we mean when we say "generalized Bohm criterion," or more specifically "kinetic Bohm criterion." For instance, the sheath criterion of equation 5.8 specifies a condition that must be satisfied at the sheath edge, yet it is not typically called a "Bohm criterion." Bohm's original criterion ($V \geq c_s$) was condition concerning the ion speed (assumed to be monoenergetic in that paper) at the sheath edge. So, an obvious thought might be that a Bohm criterion must say something about the ion speed at the sheath edge. However, it is not obvious that the previous kinetic Bohm criteria (equations 5.1 and 5.2) do this. In this section, we will look specifically for a condition concerning the fluid flow speed of ions at the sheath edge that does not depend on a spatial derivative of the fluid flow speed. The generalization that we seek over Bohm's formulation is that we do not assume monoenergetic ions and we specify the ion fluid flow speed as the $\mathbf{v}$ moment of f_s using equation 5.25.

The basic assumptions that we will make at the outset are that the plasma is in steady state (so $\partial/\partial t$ terms vanish) and that the only spatial variation in f_s is due to the electric field drive of the sheath and presheath. We take this electric field to be in the $\hat{z}$ direction. This means, for instance, that the density gradient is given by

$$\frac{d}{d\mathbf{x}} n_s = \int d^3v \left(\underbrace{\frac{\partial f_s}{\partial x}}_{=0} \hat{x} + \underbrace{\frac{\partial f_s}{\partial y}}_{=0} \hat{y} + \frac{\partial f_s}{\partial z} \hat{z} \right) = \frac{dn_s}{dz} \hat{z}. \tag{5.38}$$

Likewise, $\nabla \cdot \mathbf{V}_s = dV_{z,s}/dz$, etc.... This can also be stated as $n_s, \mathbf{V}_s, T_s, \text{etc}\ldots$ are only functions of the spatial variable z.

Recall that the sheath criterion of equation 5.8 specifies $\sum_s q_s dn_s/dz|_{z=0} \geq 0$. We can relate n_s

and V_s using the continuity equation 5.33. Applying our assumptions, the continuity equation gives

$$n_s \frac{dV_{z,s}}{dz} + V_{z,s} \frac{dn_s}{dz} = 0 \quad \Rightarrow \quad \frac{dn_s}{dz} = -\frac{n_s}{V_{z,s}} \frac{dV_{z,s}}{dz}. \tag{5.39}$$

Putting this into the sheath criterion yields

$$\sum_s q_s \frac{n_s}{V_{z,s}} \frac{dV_{z,s}}{dz}\bigg|_{z=0} \leq 0, \tag{5.40}$$

which is a condition concerning the spatial gradient of V_s at the sheath edge. We are looking for a condition on V_s itself. We can find an expression for dV_s/dz from the $\hat{z}$ component of the momentum evolution equation 5.34. Applying the aforementioned assumptions, this is

$$m_s n_s V_{z,s} \frac{dV_{z,s}}{dz} = n_s q_s E - \frac{dp_s}{dz} - \frac{d\,\Pi_{zz,s}}{dz} + R_{z,s}. \tag{5.41}$$

Putting equation 5.41 into the sheath condition from 5.40 gives the following form of a Bohm criterion

$$\boxed{\sum_s q_s \left[\frac{q_s n_s - \left(n_s\, dT_s/dz - d\Pi_{zz,s}/dz + R_{z,s} \right)/E}{m_s n_s V_{z,s}^2 - n_s T_s} \right]_{z=0} \leq 0.} \tag{5.42}$$

Equation 5.42 is a kinetic Bohm criterion because it provides a condition that the flow speed of ions must satisfy at the sheath edge (without depending on spatial gradients of V_s) and it makes no assumptions about the distribution function. It does depend on spatial gradients of higher-order moments such as the temperature and stress tensor. These could be eliminated in terms of spatial gradients of even higher order fluid moments, the heat flux in this case, by using the temperature evolution equation 5.37. However, no matter how far one carries out the hierarchy expansion, the analogous Bohm criterion will still depend on a spatial derivative of f_s inside some fluid moment integral. Equation 5.42 can be written explicitly in a kinetic form by writing the fluid variables in terms of their definitions as moments of f_s.

For the low-temperature plasmas of interest in this work, equation 5.42 simplifies to a conventional fluid result (but, where the fluid variables can still be identified in terms of their definition as velocity-space moments of f_s). This is because the terms in parenthesis in equation 5.42 are divided by E, and E is typically much bigger than these terms at the sheath edge. For example, consider the ion temperature gradient term. In the momentum balance equation 5.34, we find the scaling

$$\frac{dM_i V_i^2/dz}{dT_i/dz} \sim \frac{M_i c_s^2}{T_i} \sim \frac{T_e}{T_i} \gg 1. \tag{5.43}$$

Since $V_i^2 \sim 2\phi/M_i$, the temperature gradient term in 5.42 is small in low-temperature plasmas. The term that involves the collisional friction, R_s/E, is also negligible because the friction is typically much smaller than the electric field at the sheath edge (for the low-temperature parameters of interest here). Although it is a negligible term in equation 5.42, collisional friction can play an important role in the plasma-boundary transition. In chapter 6, we will discuss in detail the role of collisional friction in plasmas with multiple ion species.

Since the terms in parenthesis in equation 5.42 are typically negligible because of the relatively much stronger electric field at the sheath edge, equation 5.42 reduces to

$$\sum_s \frac{q_s^2\, n_{so}}{m_s V_{z,so}^2 - T_{so}} \leq 0. \tag{5.44}$$

Here the subscript o denotes that the variables are evaluated at the sheath edge ($z = 0$). Equation 5.44 can also be written in terms of f_s:

$$\sum_s \left\{ q_s^2 \left(\int d^3v\, f_s \right)^2 \Big/ \left[m_s \left(\int d^3v\, f_s \right)^{-1} \left(\int d^3v\, v_z f_s \right)^2 - \int d^3v\, \frac{1}{3} m_s v_r^2 f_s \right] \right\} \leq 0. \tag{5.45}$$

Considering a typical plasma in which the electron fluid flow speed toward the wall in the plasma boundary transition is slow compared to the electron thermal speed ($V_{z,e} \ll v_{Te}$), equation 5.44 reduces to

$$\boxed{\sum_i \frac{q_i^2}{e^2} \frac{n_{io}}{n_{eo}} \frac{c_{s,i}^2}{V_{z,i}^2 - v_{T,i}^2/2} \leq 1,} \tag{5.46}$$

in which i label the different ion species. Equation 5.46 was first derived by Riemann using a fluid approach [39]. In chapter 6, we will consider details of plasmas with more than one ion species. For plasmas with a single ion species, equation 5.46 reduces to

$$V_z \geq \sqrt{c_s^2 + v_{Ti}^2/2}. \tag{5.47}$$

Writing this explicitly in terms of the distribution functions, and applying quasineutrality ($n_i = n_e \equiv n$), yields

$$\boxed{\frac{1}{n} \int d^3v\, v_z\, f_i \geq \left[\frac{1}{3} \frac{1}{M_i} \frac{1}{n} \int d^3v\, v_r^2 \big(m_e\, f_e + M_i\, f_i \big) \right]^{1/2}.} \tag{5.48}$$

For the low temperature applications that we consider in this work, $T_e \gg T_i$, and equation 5.47 simply reduces to the usual Bohm criterion

$$V_z \geq c_s. \tag{5.49}$$

However, whereas Bohm assumed monoenergetic ions, equation 5.49 defines V_z and the T_e in $c_s = \sqrt{T_e/M_i}$ in terms of velocity-space moments of the ion and electron distribution functions (equations 5.25 and 5.29). Writing equation 5.49 explicitly in terms of these distribution functions yields

$$\frac{1}{n}\int d^3v\, v_z f_i \geq \left[\frac{1}{3}\frac{m_e}{M_i}\frac{1}{n}\int d^3v\, v_z^2 f_e\right]^{1/2}. \tag{5.50}$$

5.3 The Role of Ion-Ion Collisions in the Presheath

In chapter 4, we considered electron-electron scattering in the presheath in detail and found that it was frequent enough (because of ion-acoustic instabilities) to drive the electron distribution to a Maxwellian. We have yet to consider ion-ion scattering in the presheath. We do so now in order to gain some insight into determining f_i at the sheath edge. If f_i could be determined, the result could then be used in equation 5.50 to find a criterion that ions satisfy as they leave the plasma. Recall from equation 4.55 that, for a typical thermal particle, the scattering frequency of like-particle collisions scales as

$$\nu^{s/s} \sim \frac{C(f_s, f_s)}{f_s} \sim \frac{n_s}{m_s v_{Ts}^2}\left(\mathcal{Q}_{\mathrm{LB}}^{s/s} + \mathcal{Q}_{\mathrm{IE}}^{s/s}\right). \tag{5.51}$$

Recall also that the mass and temperature scalings of the collisional kernels are

$$\mathcal{Q}_{\mathrm{LB}}^{s/s} = \frac{2q_s^2 q_s^2}{m_s}\int d^3k \frac{\mathbf{kk}}{k^4}\frac{\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]}{\left|\hat{\varepsilon}(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})\right|^2} \sim \frac{1}{m_s v_{Ts}} \tag{5.52}$$

and

$$\mathcal{Q}_{\mathrm{IE}}^{s/s} \approx \sum_j \frac{2q_s^2 q_s^2}{m_s}\int d^3k \frac{\mathbf{kk}}{k^4}\frac{\pi\,\delta[\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}')]\,\delta(\omega_{R,j} - \mathbf{k}\cdot\mathbf{v})\exp(2\gamma_j t)}{\gamma_j\left|\partial\hat{\varepsilon}(\mathbf{k},\omega)/\partial\omega\right|^2_{\omega_j}} \sim \frac{1}{m_s v_{Ts}}. \tag{5.53}$$

Since both $\mathcal{Q}_{\mathrm{LB}}^{s/s}$ and $\mathcal{Q}_{\mathrm{IE}}^{s/s}$ have the same scaling with mass and temperature, we find

$$\frac{\nu^{i/i}}{\nu^{e/e}} \sim \frac{m_e v_{Te}^3}{M_i v_{Ti}^3} \sim \sqrt{\frac{m_e}{M_i}}\left(\frac{T_e}{T_i}\right)^{3/2}, \tag{5.54}$$

for both the Lenard-Balescu and instability-enhanced terms. In many astrophysical and fusion plasmas, $T_e \approx T_i$, so ions equilibrate with one another on a slower timescale than electrons by a factor of $\sqrt{m_e/M_i} \sim 40$. However, in the low-temperature plasmas we are interested in here, we find that ions equilibrate with one another faster than electrons do. For an electron-proton plasmas with room temperature ions and 1 eV electrons, we find

$$\frac{\nu^{i/i}}{\nu^{e/e}} \sim \sqrt{\frac{m_e}{M_i}}\left(\frac{T_e}{T_i}\right)^{3/2} \sim \left(\frac{1}{40}\right)(400) \sim 10. \tag{5.55}$$

Recall from equation 4.62 that the electron-electron scattering length in Langmuir's mercury plasma from the Lenard-Balescu contribution was about 28 cm. After instability-enhanced collisions from ion-acoustic instabilities were accounted for, this fell to approximately 0.2 cm. For Langmuir's plasma (with $T_e = 2$ eV and mercury ions) equation 5.54 gives $\nu^{i/i}/\nu^{e/e} \sim 2.5$. Thus, we expect ion-ion collisions to cause equilibration to a Maxwellian about 2 times faster than the electrons equilibrate. Using our results from section 4.4.2, we thus predict that ion-acoustic instabilities cause the ion distribution to be Maxwellian as well as the electron distribution. The electron-ion collision frequency scales as $\nu^{e/i}/\nu^{e/e} \sim m_e/M_i \sim 10^{-4}$. Thus, even with the enhanced collisions from ion-acoustic instabilities, we do not expect electrons and ions to equilibrate with one another. Because of this, ions can flow relative to the electrons and the ion-acoustic instability drive remains. It is also noteworthy that previous experimental measurements using LIF have shown that the ion distribution has a flow-shifted Gaussian shape in the presheath (see, e.g., [48]). This is consistent with our prediction that the distribution is Maxwellian.

5.4 Examples for Comparing the Different Bohm Criteria

In this section, we will consider two different ion distribution functions, flowing Maxwellian and delta function (mononenergetic), and two different electron distribution functions, stationary Maxwellian and Maxwellian with a depleted tail (for a case where collisions may not have repleted this part of the distribution). We have seen throughout chapter 4 and section 5.3 of this chapter that these are all possible, sometimes expected, distribution functions in the plasma-boundary transition (a delta function is not technically possible, but we want to look at this monoenergetic ion case in order to reduce the kinetic models in this chapter to the original problem that Bohm studied). For each of these distribution functions, we will compare the condition from the previous kinetic Bohm criterion, equation 5.1, with the condition from equation 5.48.

Monoenergetic ions, Maxwellian electrons: We will start with the idealized plasma that Bohm considered in his original paper [38]. This assumed monoenergetic ions, $f_i = n_i \delta(\mathbf{v} - \mathbf{V}_i)$, and Maxwellian electrons. Here we align coordinates so that $\mathbf{V}_i = V_i \, \hat{z}$. We saw in equation 3.60 that if equations 5.24, 5.25 and 5.29 are used to define the density, fluid flow velocity and temperature,

that this determines the five coefficients $A, \mathbf{B}$ and C in the general Maxwellian distribution $f_{Ms}(\mathbf{v}) = \exp\big(-Av^2/2 + \mathbf{B}\cdot\mathbf{v} + C\big)$. These definitions yield: $A = m_s/(2T_s)$, $\mathbf{B} = m_s\mathbf{V}_s/T_s$ and $\exp(-C) = n_s/(\pi^{3/2}v_{Ts}^3)\exp(-V_s^2/v_{Ts}^2)$. Thus, the Maxwellian can be written in the conventional form

$$f_{Ms}(\mathbf{v}) = \frac{n_s}{\pi^{3/2}v_{Ts}^3}\exp\left[-\frac{(\mathbf{v}-\mathbf{V}_s)^2}{v_{Ts}^2}\right]. \tag{5.56}$$

For the electron and ion distribution functions cited above, the components of equation 5.1 are

$$\frac{1}{M_i}\int d^3v\,\frac{f_i}{v_z^2} = \frac{1}{M_i}\int d^3v\frac{n_i\delta(\mathbf{v}-\mathbf{V})}{v_z^2} = \frac{n_i}{M_i}\frac{1}{V_i^2} \tag{5.57}$$

and

$$-\frac{1}{m_e}\int d^3v\,\frac{1}{v_z}\frac{\partial f_{Me}}{\partial v_z} = \frac{2}{m_e}\frac{1}{v_{Te}^2}\int d^3v\,\frac{v_x+v_y+v_z}{v_z}f_{Me} = \frac{2}{m_e}\frac{1}{v_{Te}^2}\int dv_z\,f_{Me} = \frac{n_e}{T_e}. \tag{5.58}$$

The components of equation 5.48 are

$$\int d^3v\,v_z f_i = \int d^3v\,v_z\,n_i\,\delta(\mathbf{v}-\mathbf{V}_i) = n_iV_i, \tag{5.59}$$

$$\frac{1}{3}\int d^3v\,v_r^2 f_i = \frac{1}{3}\int d^3v\big(v^2 - 2vV_i + V_i^2\big)f_i = \frac{1}{3}n_i\big(V_i^2 - 2V_i^2 + V_i^2\big) = 0, \tag{5.60}$$

and

$$\frac{1}{3}\int d^3v\,v_r^2\,m_e f_e = T_e. \tag{5.61}$$

Inserting equations 5.57 and 5.58, and assuming quasineutrality, the kinetic Bohm criterion derived by previous authors (equation 5.1) reduces to the conventional Bohm criterion: $V_i \geq c_s = \sqrt{T_e/M_i}$. Putting equations 5.59, 5.60 and 5.61 into equation 5.48 also gives the conventional Bohm criterion: $V_i \geq c_s$. Thus, the previously derived kinetic Bohm criterion from the literature (equation 5.1), the kinetic equation developed in section 5.2.2, and Bohm's original work [38] all provide the same criterion for plasmas with monoenergetic ions and Maxwellian electrons.

Maxwellian ions, Maxwellian electrons: Next, we consider a plasma with stationary Maxwellian electrons and flowing Maxwellian ions. From our work in chapter 4 and section 5.3 we showed that, because of the short scattering lengths for both ion-ion and electron-electron collisions in the presheath, this is a physically meaningful situation for low-temperature plasmas. In this case, equation 5.48 simply reduces to $V_i \geq \sqrt{c_s^2 + v_{Ti}^2/2}$. However, the ion term in the kinetic Bohm criterion from equation 5.1 diverges

$$\frac{1}{M_i}\int d^3v\frac{f_{Mi}(\mathbf{v})}{v_z^2} = \frac{n_i}{2\sqrt{\pi}T_i}\int_{-\infty}^{\infty}dv_z\frac{\exp(-v_z^2/v_{Ti}^2)}{v_z^2} \to \infty \tag{5.62}$$

(recall that this integral was formally shown to diverge in equation 5.19). Thus, equation 5.1 gives the condition $\infty \leq n_e/T_e$, which does not agree with the condition from equation 5.48.

Monoenergetic ions, truncated Maxwellian electrons: To show that there is also difference in the electron term of equations 5.1 and 5.48, we consider an electron distribution that is Maxwellian except for that it is truncated for some velocity in the $\hat{z}$ direction, which we denote $v_{\parallel,c}$. Our motivation here is a theoretical exercise to demonstrate the difference between equations 5.1 and 5.48. However, it can also be a physically relevant situation. If some strong damping mechanism is present, such as neutral damping, the ion-acoustic instability-enhanced driver for electron-electron collisions might be missing. In this case, the tail of the electron distribution function is expected to be depleted for energies beyond what is required to escape the sheath. The truncation velocity is in the direction parallel to the sheath electric field and is given by $v_{\parallel c} = \sqrt{2\Delta\phi_s/m_e}$. A truncated electron distribution can be written

$$f_e = \frac{\bar{n}_e}{\pi^{3/2}\bar{v}_{Te}^3}\exp\left(-\frac{v^2}{\bar{v}_{Te}^2}\right)H(v_{\parallel,c} - v_z) \tag{5.63}$$

in which H is the Heaviside step function. Note that $\bar{n}_e$ is not the density and $\bar{T}_e$ is not the electron temperature, as they are defined in equations 5.24 and 5.29. In terms of the fluid variable definitions, the density is

$$n_e = \int d^3v\, f_e = \frac{\bar{n}_e}{\sqrt{\pi}\bar{v}_{Te}}\int_{-\infty}^{v_{\parallel,c}} dv_z\, \exp\left(-\frac{v_z^2}{\bar{v}_{Te}^2}\right) = \frac{\bar{n}_e}{2}\left[1 + \mathrm{erf}\left(\frac{v_{\parallel,c}}{\bar{v}_{Te}}\right)\right], \tag{5.64}$$

the flow velocity is

$$\begin{aligned}\mathbf{V}_e &= \frac{1}{n_e}\int d^3v\, \mathbf{v} f_e = \frac{1}{n_e}\frac{\bar{n}_e}{\sqrt{\pi}\bar{v}_{Te}}\int_{-\infty}^{v_{\parallel,c}} dv_z\, v_z \exp\left(-\frac{v_z^2}{\bar{v}_{Te}}\right)\hat{z} = -\frac{1}{2\sqrt{\pi}}\frac{\bar{n}_e}{n_e}\exp\left(-\frac{v_{\parallel,c}^2}{\bar{v}_{Te}^2}\right)\bar{v}_{Te}\,\hat{z} \qquad (5.65)\\ &= -\frac{1}{\sqrt{\pi}}\frac{\exp\left(-v_{\parallel,c}^2/v_{Te}^2\right)}{1+\mathrm{erf}\left(v_{\parallel,c}/\bar{v}_{Te}\right)}\bar{v}_{Te}\,\hat{z}\end{aligned}$$

and the temperature is

$$\begin{aligned}T_e &= \frac{1}{n_e}\int d^3v\,\frac{1}{3}m_e\, v_r^2 f_e = -\frac{1}{3}m_e V_e^2 + \frac{1}{3}\frac{m_e}{n_e}\int d^3v\, v^2 f_e = \frac{1}{3}m_e V_e\left(v_{\parallel,c} - V_e\right) + \frac{1}{2}m_e\bar{v}_{Te}^2 \qquad (5.66)\\ &= \frac{1}{2}m_e\bar{v}_{Te}^2\left[1 - \frac{3}{2\pi}\frac{\exp\left(-2v_{\parallel,c}^2/v_{Te}^2\right)}{\left[1+\mathrm{erf}\left(v_{\parallel,c}/\bar{v}_{Te}\right)\right]^2} - \frac{3}{2\sqrt{\pi}}\frac{v_{\parallel,c}}{\bar{v}_{Te}}\frac{\exp\left(-v_{\parallel,c}^2/\bar{v}_{Te}^2\right)}{\left[1+\mathrm{erf}\left(v_{\parallel,c}/\bar{v}_{Te}\right)\right]}\right].\end{aligned}$$

For this example, equation 5.48 reduces to the following Bohm criterion

$$V_i \geq \left\{\frac{1}{2}\frac{m_e}{M_i}\bar{v}_{Te}^2\left[1 - \frac{3}{2\pi}\frac{\exp\left(-2v_{\parallel,c}^2/v_{Te}^2\right)}{\left[1+\mathrm{erf}\left(v_{\parallel,c}/\bar{v}_{Te}\right)\right]^2} - \frac{3}{2\sqrt{\pi}}\frac{v_{\parallel,c}}{\bar{v}_{Te}}\frac{\exp\left(-v_{\parallel,c}^2/\bar{v}_{Te}^2\right)}{\left[1+\mathrm{erf}\left(v_{\parallel,c}/\bar{v}_{Te}\right)\right]}\right]\right\}^{1/2}. \tag{5.67}$$

The electron term in the Bohm criterion from equation 5.1, is given by

$$-\frac{1}{m_e}\int d^3v\,\frac{1}{v_z}\frac{\partial f_e}{\partial v_z} = -\frac{1}{m_e}\frac{\bar{n}_e}{\pi^{3/2}\bar{v}_{Te}^3}\left[\int d^3v\frac{-2(v_x+v_y+v_z)}{\bar{v}_{Te}^2 v_z}\exp\left(-\frac{v^2}{\bar{v}_{Te}^2}\right)H(v_{\parallel,c}-v_z)\right. \tag{5.68}$$
$$\left. -\int d^3v\frac{\delta(v_{\parallel,c}-v_z)}{v_z}\exp\left(-\frac{v_z^2}{\bar{v}_{Te}^2}\right)\right]$$
$$= \frac{\bar{n}_e}{m_e\bar{v}_{Te}^2}\left[1+\mathrm{erf}\left(\frac{v_{\parallel,c}}{\bar{v}_{Te}}\right)+\frac{1}{\sqrt{\pi}}\frac{\bar{v}_{Te}}{v_{\parallel,c}}\exp\left(-\frac{v_{\parallel,c}^2}{\bar{v}_{Te}^2}\right)\right].$$

With this, equation 5.1 give the condition

$$V_i \geq \left\{\frac{1}{2}\frac{m_e}{M_i}\bar{v}_{Te}^2\left[\frac{1+\mathrm{erf}\left(v_{\parallel,c}/\bar{v}_{Te}\right)}{1+\mathrm{erf}\left(v_{\parallel,c}/\bar{v}_{Te}\right)+\frac{v_{Te}}{\sqrt{\pi}v_{\parallel,c}}\exp\left(-v_{\parallel,c}^2/\bar{v}_{Te}^2\right)}\right]\right\}^{1/2}. \tag{5.69}$$

Equations 5.67 and 5.69 give different results (although, they converge to the same result as $v_{\parallel,c}/v_{Te}\rightarrow\infty$). For example, consider the case $v_{\parallel,c}=0$. In this case, equation 5.67 gives

$$V_i \geq \left[\frac{1}{2}\frac{m_e}{M_i}\bar{v}_{Te}^2\left(1-\frac{3}{2\pi}\right)\right]^{1/2}, \tag{5.70}$$

but the kinetic criterion of equation 5.1 gives

$$\frac{n_i}{M_i}\frac{1}{V_i^2}\geq\infty. \tag{5.71}$$

Thus, not only does the ion term have divergence issues, but also the electron term in the kinetic Bohm criterion from equation 5.1. The approach of section 5.2.2 corrects these divergence issues.

Chapter 6

Determining the Bohm Criterion In Multiple-Ion-Species Plasmas

Understanding plasma-boundary interactions requires knowing the speed at which ions leave a plasma. Determining this speed is important in a broad range of plasma applications. For example, the speed that ions fall into a sheath determines the depth and anisotropy of tunnels in plasma etching of semiconductors [80], the depth and flux of ions at a surface in plasma-based ion implantation [96] and the flux and speed of ions required to interpret Langmuir probe measurements [97]. Other examples include determining the flux, heat load and recycling rates at boundaries in the scrape-off layer of fusion experiments [98], and the interaction of ionospheric or interstellar plasmas with spacecraft [99]. In all of these examples multiple species of positive ions are often present.

We showed in equation 5.46 of section 5.2.2 that the Bohm criterion

$$\sum_i \frac{q_i^2}{e^2} \frac{n_{io}}{n_{eo}} \frac{c_{s,i}^2}{V_{z,i}^2 - v_{T,i}^2/2} \leq 1 \tag{6.1}$$

provides a condition that the ion flow speed must satisfy at the sheath edge (which is the boundary of a quasineutral plasma). Equation 6.1 was first derived by Riemann [39]. If the plasma contains a single species of positive ions that are cold compared to the electrons, equation 6.1 reduces to the conventional Bohm criterion: $V \geq c_s$. It has been shown theoretically [81], and experimentally [84], that equality typically holds in the conventional Bohm criterion. In this case, $V = c_s$ uniquely determines the ion flow speed at the sheath edge. We assume that equality holds in equation 6.1 as well. However, even if equality holds, equation 6.1 does not uniquely determine the flow speed of each ion species at the sheath edge if more than one ion species is present. To determine which of the infinite number of possible solutions is physically realized, is what we mean by "determining the Bohm criterion." Determining

the Bohm criterion in multiple-ion-species plasmas will be the subject of this chapter. We will show that collisional friction between ion species due to instability-enhanced interactions, which arise from two-stream instabilities in the presheath, often plays a critical role in this determination.

6.1 Previous Work on Determining the Bohm Criterion in Two-Ion-Species Plasmas

Because it is important in so many applications, a significant amount of literature exists on determining the Bohm criterion in multiple-ion-species plasmas. Almost all of this work is concerned with plasmas with two species of positively charged ions and electrons. This is the situation that we will concentrate on as well. Plasmas with negative ion species are also discussed in the literature, but this topic will not be a focus of the present work. The major issue we address in this chapter is why the previous theories [43–47] and experiments [48–52, 100–102] do not agree as to what flow speed each ion species has as it leaves the plasma. Both the theory and experiments in this area concentrate on low ion temperature plasmas (which satisfy $T_i \ll T_e$) with ions that have a single positive charge. In this case equation 6.1 reduces to

$$\sum_i \frac{n_{io}}{n_{eo}} \frac{c_{s,i}^2}{V_{io}^2} \leq 1. \tag{6.2}$$

The vast majority of theoretical work on this topic has been published by Franklin [43–47]. Using a variety of analytic and computational models, Franklin predicts that each ion species should fall into the sheath with a speed close to its individual sound speed: $V_i = c_{s,i}$, where the individual sound speed is defined as

$$c_{s,i} \equiv \sqrt{\frac{T_e}{M_i}}. \tag{6.3}$$

It can easily be confirmed that this is one possible solution of the Bohm criterion in equation 6.2. Franklin's model consists of a set of fluid continuity and momentum balance equations that account for ion-neutral collision processes and ionization sources, in addition to the usual plasma physics terms. His model does not include the effect of ion-ion friction. We will see in section 6.3 that in stable plasmas this is a valid approximation. It is not valid if ion-ion two-stream instabilities are present. By solving this set of equations throughout the plasma-boundary transition region, Franklin finds that each ion species

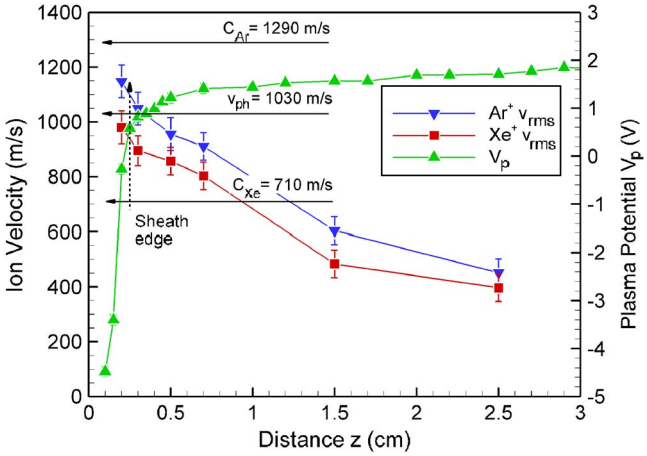

Figure 6.1: LIF data showing the ion fluid speeds of Ar+ (blue triangles) and Xe+ (red squares) throughout the presheath (left axis) and the electrostatic potential profile (right axis, green diamonds). Also indicated are the individual and system sound speeds. The density of each ion species was approximately the same in this experiment. This figure has been reproduced from reference [50]. Copyright 2007 by The American Institute of Physics.

tends to fall into the sheath with a speed near its individual sound speed $c_{s,i}$. Deviations from this solution can occur when there is a lot of ion-neutral drag. Particularly if there is much more ion-neutral drag on one ion species than the other. However, for common plasma parameters these deviations tend to be small and the individual sound speed solution is a robust prediction of these model equations.

Experiments measure a significantly different speed for each ion species than their individual sound speeds at the sheath edge [48–52, 100–102]. Most of these experiments employed the laser-induced fluorescence (LIF) technique [48–50, 100, 102] and directly measured the speed of each ion species as it traversed the presheath and entered the sheath. This work measured the individual ion flow speeds and showed that they tended to be much closer to a common speed at the sheath edge than their individual sound speeds. This common speed was the system sound speed $V_i = c_s$, which is a density-weighted average of the individual sound speeds

$$c_s \equiv \sqrt{\sum_i \frac{n_i}{n_e} c_{s,i}^2}. \tag{6.4}$$

This common sound speed is the solution of equation 6.2 if one assumes that $V_1 = V_2$.

Additional experimental evidence for the common sound speed solution has been provided using a

combination of electrostatic probes and ion-acoustic waves [51, 52] to measure the ion speeds in the plasma-boundary transition region. Oksuz *et al* [52] have shown empirically that, for two ion species plasmas, the ion acoustic wave speed at the sheath edge is typically twice what it is in the bulk plasma. Taking this observation as an ansatz, Lee *et al* [53] have shown that it implies each ion species enters the sheath at the common system sound speed. However, no physical mechanism has been suggested by which this solution is established.

The majority of experiments that have been reported used either Ar-Xe or Ar-He plasma in which the density of each ion species was approximately equal (and, thus, half of the electron density). When considering a specific example plasma in this chapter, we will use the parameters of a particularly well diagnosed plasma from the literature [50]. This was a Ar-Xe plasma in which each ion species had approximately the same density, the electron density was 5×10^9 cm^{-3}, the electron temperature was 0.7 eV and the ion temperature was approximately room temperature (0.02 eV). The figure from Lee *et al* [50] that presents LIF data for the flow speed of each ion species throughout the plasma-boundary transition in this plasma has been reproduced here in figure 6.1.

The fact that experiments have measured the individual ion flow speeds to be much closer to one another than the theoretical models predict suggests that ion-ion friction between the species might be important. However, if one calculates the expected ion-ion friction from Coulomb interactions in a stable plasma, it turns out to be a weak effect (assuming the low-temperature plasma parameters from the experiment). This calculation is shown in section 6.3. It appears that some mechanism other than the conventional Coulomb collisions in stable plasma must be present in order to explain the experimental measurements.

In this chapter, we show that the physical mechanism responsible for enhancing the collisional friction between each ion species is ion-ion two-stream instabilities. In plasmas with $T_e \gg T_i$, two-stream instabilities can grow in the presheath when any two ion species have speeds that differ by more than a critical value that is characteristic of their thermal speeds. These instabilities greatly enhance the collisional friction between each ion species. It causes the difference in their flow speeds to become fixed to a value that can be far less than the difference of their individual sound speeds. In the limit of vanishing ion temperatures, both species are predicted to enter the sheath at a common system sound speed c_s.

Ion-ion two stream instabilities have been measured in the presheath in previous experiments [49, 51]. These references show measurements of broad-band noise (significantly above the thermal level) in the MHz frequency range near the plasma boundaries in Ar-He plasma. They also show that the instability is strongest when the relative concentration of each ion species is similar, and that the instabilities become much weaker when the concentration of one species is much more, or less, than the other. All of these results are consistent with the two-stream instabilities that we discuss in section 6.4. This problem of determining Bohm's criterion in multiple-ion-species plasmas has a lot in common with the Langmuir's paradox problem that we discussed in chapter 4. Both problems are concerned with a measurement that appears to show anomalous scattering amongst particles, and in both problems instabilities have been measured. However, previous theories could not show how the measured instabilities could explain the anomalous effect (either Maxwellian electron distribution functions or a collisional friction between two particular species). Again we can bridge this gap by applying the theory developed in chapter 2.

6.2 Momentum Balance Equation and the Frictional Force

Recall from equation 5.34 that the momentum balance equation is given by

$$m_s n_s \left(\frac{\partial \mathbf{V}_s}{\partial t} + \mathbf{V}_s \cdot \frac{\partial \mathbf{V}_s}{\partial \mathbf{x}} \right) = n_s q_s \big(\mathbf{E} + \mathbf{V}_s \times \mathbf{B} \big) - \frac{\partial p_s}{\partial \mathbf{x}} - \frac{\partial}{\partial \mathbf{x}} \cdot \Pi_s + \mathbf{R}_s. \tag{6.5}$$

In this chapter, we will be interested in the collisional friction force density

$$\mathbf{R}_s(\mathbf{x}, t) \equiv \int d^3v \; m_s \mathbf{v} C(f_s). \tag{6.6}$$

Although it is a force density, we will simply refer to this as the collisional friction. Since the collision operator is the sum of the Lenard-Balescu term and the instability-enhanced term $C(f_s) = C_{\mathrm{LB}}(f_s) + C_{\mathrm{IE}}(f_s)$, the collisional friction can be written as the sum of the two contributions: $\mathbf{R}_s = \mathbf{R}_{\mathrm{LB},s} + \mathbf{R}_{\mathrm{IE},s}$. Noting that the total collision operator can be written in terms of component collision operators $C(f_s) = \sum_{s'} C(f_s, f_{s'})$, the collisional friction can also be written in terms of component contributions $\mathbf{R}_s = \sum_{s'} \mathbf{R}^{s-s'}$. This property is obeyed by both the Lenard-Balescu collision operator and the instability-enhanced collision operator derived in chapter 2 (and, hence, also the associated collisional friction terms), but it is not obeyed by previous theories of wave-particle scattering such as quasilinear

theory; see section 3.4. This property will be essential here because we are only interested in the friction between ion species 1 and 2: $\mathbf{R}^{1-2} = \mathbf{R}^{1-2}_{\mathrm{LB}} + \mathbf{R}^{1-2}_{\mathrm{IE}}$.

Another property that will be important in this chapter is that the frictional force between individual species is equal and opposite

$$\mathbf{R}^{s-s'} = -\mathbf{R}^{s'-s}. \tag{6.7}$$

This is a direct consequence of the property of conservation of momentum between individual species that was proved in section 3.4.2 (written in equation 3.38). Again, this property is not obeyed by previous theories of wave-particle scattering, such as quasilinear theory. These are two properties of the theory derived in chapter 2 that will be essential for the application considered in this chapter.

The collisional friction can be written in a more convenient form for calculation than it is in equation 6.6. First, recall that

$$\mathbf{R}^{s-s'} = \int d^3v\, m_s \mathbf{v} C(f_s, f_{s'}) = -m_s \int d^3v\, \mathbf{v}\, \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{J}^{s-s'}_v, \tag{6.8}$$

in which $\mathbf{J}^{s-s'}_v$ is the collisional current. Recalling the diad (tensor) identity $\nabla \cdot (\mathbf{AB}) = (\nabla \cdot \mathbf{A})\mathbf{B} + (\mathbf{A} \cdot \nabla)\mathbf{B}$, the integrand can be written as

$$\mathbf{v}\left(\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{J}^{s-s'}_v\right) = \frac{\partial}{\partial \mathbf{v}} \cdot \left(\mathbf{v}\, \mathbf{J}^{s-s'}_v\right) - \left(\mathbf{J}^{s-s'}_v \cdot \frac{\partial}{\partial \mathbf{v}}\right)\mathbf{v} = \frac{\partial}{\partial \mathbf{v}} \cdot \left(\mathbf{v}\, \mathbf{J}^{s-s'}_v\right) - \mathbf{J}^{s-s'}_v. \tag{6.9}$$

Putting this into equation 6.8 and applying Gauss's flux theorem yields

$$\mathbf{R}^{s-s'} = -m_s \int d^3v \frac{\partial}{\partial \mathbf{v}} \cdot \left(\mathbf{v}\, \mathbf{J}^{s-s'}_v\right) + m_s \int d^3v\, \mathbf{J}^{s-s'}_v = -m_s \oint d\mathbf{S} \cdot \left(\mathbf{v}\, \mathbf{J}^{s-s'}_v\right) + m_s \int d^3v\, \mathbf{J}^{s-s'}_v. \tag{6.10}$$

Since $\mathbf{J}^{s-s'}_v$ vanishes at the boundary at infinity in velocity space, a convenient result emerges

$$\mathbf{R}^{s-s'} = m_s \int d^3v\, \mathbf{J}^{s-s'}_v. \tag{6.11}$$

Using the $\mathbf{J}^{s-s'}_v$ calculated in chapter 2, equation 2.33, yields

$$\mathbf{R}^{s-s'} = m_s \int d^3v \int d^3v' \mathcal{Q} \cdot \left(\frac{f_s(\mathbf{v})}{m_{s'}} \frac{\partial f_{s'}(\mathbf{v}')}{\partial \mathbf{v}'} - \frac{f_{s'}(\mathbf{v}')}{m_s} \frac{\partial f_s(\mathbf{v})}{\partial \mathbf{v}}\right). \tag{6.12}$$

In this chapter we will assume that both ion species have Maxwellian distributions, with flow speeds that can be different in magnitude, but in the same direction. Applying the assumption that both s and s' are Maxwellian, equation 6.12 can be written as

$$\mathbf{R}^{s-s'} = -m_s \int d^3v \int d^3v' f_s(\mathbf{v}) f_{s'}(\mathbf{v}') \mathcal{Q} \cdot \left(\frac{\mathbf{v}' - \mathbf{V}_{s'}}{T_{s'}} - \frac{\mathbf{v} - \mathbf{V}_s}{T_s}\right). \tag{6.13}$$

Furthermore, we assume that the temperature of each ion species is approximately the same $T_s \approx T_{s'}$. Using this, along with the property $\mathcal{Q} \cdot (\mathbf{v} - \mathbf{v}') = 0$, equation 6.13 reduces to

$$\mathbf{R}^{s-s'} = -\frac{m_s}{T_s} \int d^3v \int d^3v' f_s(\mathbf{v}) f_{s'}(\mathbf{v}') \mathcal{Q} \cdot (\mathbf{V}_s - \mathbf{V}_{s'}). \tag{6.14}$$

Recall from equations 2.47 and 3.64 that $\mathcal{Q}_{\mathrm{IE}} \cdot (\mathbf{v} - \mathbf{v}') = 0$ only when the instabilities are slowly growing: $\gamma_j/\omega_{R,j} \ll 1$. This will be true of the two-stream instabilities we consider in this chapter. In the following two sections we use equation 6.14 as a starting point from which we calculate the collisional friction force density in stable and ion-ion two-stream unstable plasma.

6.3 Ion-Ion Collisional Friction in Stable Plasma

In this section we calculate the stable plasma contribution to the ion-ion collisional friction force density, $\mathbf{R}^{s-s}$, starting from equation 6.14. For this, we need the Lenard-Balescu collisional kernel

$$\mathcal{Q}_{\mathrm{LB}}^{s/s'} = \frac{2q_s^2 q_{s'}^2}{m_s} \int d^3k \frac{\mathbf{kk}\delta[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')]}{k^4 |\hat{\varepsilon}(\mathbf{k}, \mathbf{k} \cdot \mathbf{v})|^2}. \tag{6.15}$$

The characteristic $\mathbf{v}$ in the dielectric function is of the order the ion flow speed, which can be as large as the ion sound speed. Using this characteristic $\mathbf{v}$, the dielectric function can be shown to be approximately adiabatic $\hat{\varepsilon}(\mathbf{k}, \mathbf{k} \cdot \mathbf{v}) \approx 1 + 1/k^2\lambda_{De}^2$. We showed in section 1.1.5 that in this case, the Lenard-Balescu collisional kernel reduces to the Landau collisional kernel [1]

$$\mathcal{Q}_{\mathrm{LB}}^{s/s'} = \frac{2\pi q_s^2 q_{s'}^2}{m_s} \frac{u^2 I - \mathbf{uu}}{u^3} \ln \Lambda. \tag{6.16}$$

Recall that $\mathbf{u} \equiv \mathbf{v} - \mathbf{v}'$.

We choose a cylindrical coordinate system for $\mathbf{u}$ such that $\mathbf{u} = u_x\hat{x} + u_y\hat{y} + u_z\hat{z}$ where

$$u_x = u_\perp \cos\psi \quad , \quad u_y = u_\perp \sin\psi \quad \text{and} \quad u_z = u_\parallel \hat{z}. \tag{6.17}$$

We align this so that the parallel direction ($\hat{z}$) is along $\Delta\mathbf{V} \equiv \mathbf{V}_s - \mathbf{V}_{s'}$. Applying this convention, and putting the Landau collisional kernel from equation 6.16 into equation 6.14, yields

$$\mathbf{R}^{s-s'} = -\frac{2\pi q_s^2 q_{s'}^2}{T_s} \Delta\mathbf{V} \ln\Lambda \int d^3v \int d^3v' f_s(\mathbf{v}) f_{s'}(\mathbf{v}') \frac{u_\parallel}{u^3} \left(-u_\perp \cos\psi\hat{x} - u_\perp \sin\psi\hat{y} + \frac{u_\perp^2}{u_\parallel}\hat{z} \right). \tag{6.18}$$

In terms of $\mathbf{v}$ and $\mathbf{v}'$, this is

$$\mathbf{R}^{s-s'} = -\frac{2\pi q_s^2 q_{s'}^2}{T_s}\Delta\mathbf{V}\ln\Lambda\int d^3v\int d^3v' \frac{f_s(\mathbf{v})f_{s'}(\mathbf{v}')}{\left(1+\frac{(v_x-v_x')^2+(v_y-v_y')^2}{(v_z-v_z')^2}\right)^{3/2}}\times \\ \times\left(-\frac{v_x-v_x'}{(v_z-v_z')^2}\hat{x}-\frac{v_y-v_y'}{(v_z-v_z')^2}\hat{y}+\frac{(v_x-v_x')^2+(v_y-v_y')^2}{(v_z-v_z')^3}\hat{z}\right). \tag{6.19}$$

Recall our assumption that the flow of each ion species is only in the $\hat{z}$ direction. With this, the Maxwellian distribution function of each species has the form

$$f_s(\mathbf{v}) = \frac{n_s}{\pi^{3/2}v_{Ts}^3}\exp\left(-\frac{v_x^2+v_y^2+v_z^2+V_s^2-2v_zV_s}{v_{Ts}^2}\right), \tag{6.20}$$

and

$$f_{s'}(\mathbf{v}') = \frac{n_{s'}}{\pi^{3/2}v_{Ts'}^3}\exp\left(-\frac{v_x'^2+v_y'^2+v_z'^2+V_{s'}^2-2v_z'V_{s'}}{v_{Ts'}^2}\right). \tag{6.21}$$

Next, we show that the $\hat{x}$ and $\hat{y}$ components of $\mathbf{R}_{\mathrm{LB}}^{s-s'}$ will vanish. It will suffice to just consider the $\hat{x}$ direction since the $\hat{y}$ component follows analogously by simply replacing x subscripts with y subscripts. Considering the v_x and v_x' integrations from equation 6.19, we have

$$\mathbf{R}_x \propto \int dv_x\int dv_x'\frac{e^{-v_x^2/v_{Ts}^2}e^{-v_x'^2/v_{Ts'}^2}(v_x-v_x')}{\left[a+(v_x-v_x')^2\right]^{3/2}} = \int du_x\frac{u_x}{(a+u_x^2)^{3/2}}\int dv_x' e^{-v_x'^2/v_{Ts'}^2}e^{-(u_x+v_x')^2/v_{Ts}^2} \tag{6.22}$$

in which we have substituted $u_x = v_x - v_x'$ and $a \equiv (v_y-v_y')^2+(v_z-v_z')^2$. Noting that

$$\int_{-\infty}^{\infty} dx e^{-\alpha x^2}e^{-\beta(x+y)^2} = \frac{\sqrt{\pi}}{\sqrt{\alpha+\beta}}\exp\left(-\frac{\alpha\beta y^2}{\alpha+\beta}\right) \tag{6.23}$$

we find that the $\hat{x}$ component of $\mathbf{R}_{\mathrm{LB}}^{s-s'}$ is given by

$$\mathbf{R}_x \propto \int_{-\infty}^{\infty} du_x\frac{u_x}{(a+u_x^2)^{3/2}}\exp\left(-\frac{\alpha\beta u_x^2}{\alpha+\beta}\right) = 0, \tag{6.24}$$

which vanishes due to odd parity of the integrand. Hence, the perpendicular components of the frictional force vanish as expected. Here we have used the definitions $\alpha \equiv 1/v_{Ts'}^2$ and $\beta \equiv 1/v_{Ts}^2$.

Now we turn to the important component – along the flow. For this, we have

$$\mathbf{R}_{\mathrm{LB}}^{s-s'} = -\frac{2\pi q_s^2 q_{s'}^2}{T_s}\Delta\mathbf{V}\ln\Lambda\int d^3v\int d^3v' f_{s'}(\mathbf{v}')f_s(\mathbf{v})\frac{u_\perp^2}{u^3}. \tag{6.25}$$

Switching the v variables to u, we find

$$\mathbf{R}_{\mathrm{LB}}^{s-s'} = -\frac{2q_s^2q_{s'}^2}{T_s}\frac{n_sn_{s'}\ln\Lambda\,\Delta\mathbf{V}}{\pi^2v_{Ts}^3v_{Ts'}^3}\ I \tag{6.26}$$

in which we have defined the integral

$$I \equiv \int d^3u \frac{u_\perp^2}{u^3} \int dv_z' e^{-\alpha[v_z'+(u_z-V_s)]^2} e^{-\beta(v_z'-V_{s'})^2} \int dv_y' e^{-\alpha(v_y'+u_y)^2} e^{-\beta v_y'^2} \int dv_x' e^{-\alpha(v_x'+u_x)^2} e^{-\beta v_x'^2}, \tag{6.27}$$

and again have used $\alpha \equiv 1/v_{Ts}^2$ and $\beta \equiv 1/v_{Ts'}^2$. Applying the integral identity

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha(x+z)^2} e^{-\beta(x+y)^2} = \frac{\sqrt{\pi}}{\sqrt{\alpha+\beta}} \exp\left[-\frac{\alpha\beta(z-y)^2}{\alpha+\beta}\right] \tag{6.28}$$

the I term of equation 6.26 reduces to

$$I = \pi^{3/2} \frac{v_{Ts}^3 v_{Ts'}^3}{\bar{v}_T^3} \int d^3u \frac{u_\perp^2}{u^3} \exp\left[-\frac{u_\perp^2 + (u_\parallel - \Delta V)^2}{\bar{v}_T^2}\right] \tag{6.29}$$

in which we have defined

$$\bar{v}_T^2 \equiv v_{Ts}^2 + v_{Ts'}^2. \tag{6.30}$$

Here $\Delta V = |\Delta \mathbf{V}|$.

The azimuthal part of the u integral in equation 6.29 is trivial to evaluate since the integrand does not depend on it. The $u_\perp$ component can be evaluated with the integral

$$\int_0^\infty du_\perp \frac{u_\perp^3}{(u_\parallel^2 + u_\perp^2)^{3/2}} \exp\left(-\frac{u_\perp^2}{\bar{v}_T^2}\right) = |u_\parallel| \left[-1 + \frac{\bar{v}_T}{2|u_\parallel|}\sqrt{\pi}\left(1 + 2\frac{u_\parallel^2}{\bar{v}_T^2}\right) \exp\left(\frac{u_\parallel^2}{\bar{v}_T^2}\right) \mathrm{erfc}\left(\frac{|u_\parallel|}{\bar{v}_T}\right)\right]. \tag{6.31}$$

Putting these two integrals into equation 6.29 yields

$$I = 2\pi^{5/2} \frac{v_{Ts}^3 v_{Ts'}^3}{\bar{v}_T^3} \Bigg\{ \underbrace{\frac{\sqrt{\pi}}{2} \bar{v}_T e^{-\Delta V^2/\bar{v}_T^2} \int_{-\infty}^{\infty} du_\parallel \left(1 + 2\frac{u_\parallel^2}{\bar{v}_T^2}\right) \exp\left(\frac{2\Delta V u_\parallel}{\bar{v}_T^2}\right) \mathrm{erfc}\left(\frac{|u_\parallel|}{\bar{v}_T}\right)}_{I_1} \tag{6.32}$$

$$- \underbrace{\int_{-\infty}^{\infty} du_\parallel |u_\parallel| \exp\left[-\frac{(u_\parallel - \Delta V)^2}{\bar{v}_T^2}\right]}_{I_2} \Bigg\}.$$

The last term in equation 6.32 can be evaluated by splitting the limits of integration for the positive and negative intervals and using integration by parts to give, $I_2 = \bar{v}_T \Delta V \mathrm{erf}(\Delta V/\bar{v}_T) + \bar{v}_T^2 \exp(-\Delta V^2/\bar{v}_T^2)$. The first integral in equation 6.32 can be written in terms of cosh

$$I_1 = \sqrt{\pi}\bar{v}_T^2 \exp\left(-\frac{\Delta V^2}{\bar{v}_T^2}\right) \int_0^\infty dy\, \left(1 + 2y^2\right) \cosh\left(2\frac{\Delta V}{\bar{v}_T} y\right) \mathrm{erfc}(y), \tag{6.33}$$

which can then be written in terms of algebraic terms and erf functions by applying integration by parts several times. Evaluating I_1 and adding it to I_2, we find

$$I = \frac{\pi^3 v_{Ts}^3 v_{Ts'}^3 \bar{v}_T^2}{\Delta V^3} \left[\mathrm{erf}\left(\frac{\Delta V}{\bar{v}_T}\right) - \frac{2}{\sqrt{\pi}} \frac{\Delta V}{\bar{v}_T} \exp\left(-\frac{\Delta V^2}{\bar{v}_T^2}\right)\right] = \frac{\pi^3 v_{Ts}^3 v_{Ts'}^3 \bar{v}_T^2}{\Delta V^3} \psi\left(\frac{\Delta V^2}{\bar{v}_T^2}\right). \tag{6.34}$$

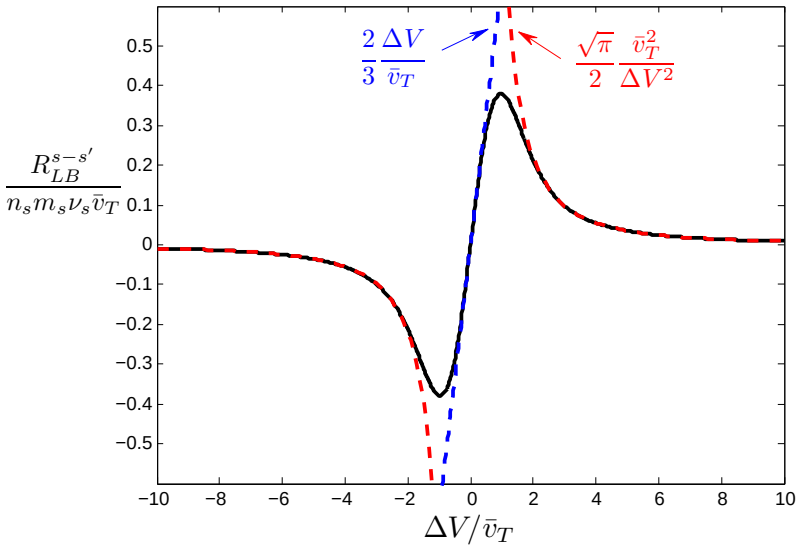

Figure 6.2: Collisional friction force density in a stable plasma between flowing Maxwellian species with the same temperature (solid black line). The blue dashed line represents the lowest order of the conventional Spitzer result for flows slow compared to thermal speeds [103], and the red dashed line is the asymptotic expansion for flows fast compared to thermal speeds.

in which we have identified the Maxwell integral:

$$\psi(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x dt\, \sqrt{t} e^{-t} = \mathrm{erf}(\sqrt{x}) - \frac{2}{\sqrt{\pi}} \sqrt{x} e^{-x}. \tag{6.35}$$

Putting equation 6.34 into equation 6.26 gives an expression for the frictional force density between two Maxwellian species, with the same temperatures, in a stable plasma

$$\boxed{\mathbf{R}_{\mathrm{LB}}^{s-s'} = -\frac{\sqrt{\pi}}{2} n_s m_s \nu_s \frac{\bar{v}_T^3 \, \Delta \mathbf{V}}{\Delta V^3} \, \psi\left(\frac{\Delta V^2}{\bar{v}_T^2}\right).} \tag{6.36}$$

in which

$$\boxed{\nu_s \equiv \frac{8\sqrt{\pi} q_s^2 q_{s'}^2 n_{s'} \ln \Lambda}{m_s^2 v_{Ts}^2 \bar{v}_T}} \tag{6.37}$$

is a reference collision frequency.

To connect with previous theories, like the classic work of Spitzer [103], and to check that equation 6.36 reduces to an established result in the appropriate limit, consider the limit $\Delta V/\bar{v}_T \ll 1$. That is, a flow difference that is small compared to the average thermal speed. In this case, we apply the small

argument series expansion of ψ and find

$$\mathbf{R}_{\mathrm{LB}}^{s-s'} \approx -n_s m_s \frac{2}{3}\nu_s \Delta\mathbf{V}. \tag{6.38}$$

The Spitzer problem considers electrons slowing on ions with $T_e \approx T_i$ such that the flow is small compared to the electron thermal speed [103]. In this case $v_{Te}^2 + v_{Ti}^2 \approx v_{Te}^2$, and $(V_e - V_i)/v_{Te} \ll 1$. Equation 6.36 predicts for this limit, $\mathbf{R}_{e/i} = n_e m_e \nu_e (\mathbf{V}_e - \mathbf{V}_i)$, where the reference electron collision frequency is

$$\nu_e = \frac{16\pi}{3\sqrt{\pi}} \frac{n_i Z^2 e^4 \ln\Lambda}{m_e^2 v_{Te}^3}. \tag{6.39}$$

This returns the Spitzer collision frequency (which leads to the Spitzer resistivity) [103]. This result is consistent with the lowest order $(0,0)$ component (where both functions are Maxwellians) of the more general Spitzer problem, which considers deviations from Maxwellian as higher-order components.

For the presheath problem, we are interested the limit where the relative flow speed is much faster than the thermal speed $\Delta V/\bar{v}_T \gg 1$, which is the opposite limit as the Spitzer problem. Using the asymptotic expansion for large argument in the ψ function, we find

$$\mathbf{R}_{\mathrm{LB}}^{s-s'} \approx -n_s m_s \frac{\sqrt{\pi}}{2}\nu_s \frac{\bar{v}_T^3}{\Delta V^3}\Delta\mathbf{V} \tag{6.40}$$

in this limit.

Equation 6.36 is plotted in figure 6.2 along with the asymptotic and power series expansions from equations 6.38 and 6.40. For our example plasma parameters, this stable plasma contribution to the collisional friction force density is much smaller than other terms in the momentum balance equation 6.5. For example, the $V_i dV_i/dz$ term in the momentum equation is much larger than $\mathbf{R}_{\mathrm{LB}}$: $\mathcal{O}[(c_s^2/\lambda^{i/n})/R_{\mathrm{LB}}^{1-2}] \sim 10^{-1}$. Thus, the neglect of ion-ion collisional friction in stable plasma is justified in Franklin's [43–47] previous theoretical work. However, it is not so small that one could claim that conventional Coulomb interactions are never important in the ion dynamics of these plasmas. It happens that it does not seem to be a significant effect for the example plasma parameters from reference [50]. We next turn to calculating contributions to the collisional friction that can come about from instability-enhanced collisions when two-stream instabilities arise in the presheath.

6.4 Ion-Ion Collisional Friction in Two-Stream Unstable Plasma

In this section, we calculate the ion-ion collisional friction force when it is enhanced by two-stream instabilities. Two-stream instabilities require that the difference in flow speeds exceed a critical value (ΔV_c) before they become unstable: $\mathbf{V}_1 - \mathbf{V}_2 \equiv \Delta\mathbf{V} > \Delta\mathbf{V}_c$. This critical speed is characteristic of the ion thermal speed: $\Delta V_c \sim \mathcal{O}(v_{Ti})$. In section 6.4.2, we calculate the instability-enhanced collisional friction from two-stream instabilities in the presheath. We find that whenever the two-stream instabilities are present, the collisional friction quickly (within a few Debye lengths) becomes so large that it dominates the momentum balance equation. This creates a very stiff system whereby if two-stream instabilities are present, the resultant friction quickly forces the flow speed of each species together (this is because the frictional force between them is equal and opposite; see equation 6.7).

Since the friction force quickly dominates whenever these instabilities arise, the difference in flow speeds can never exceed the critical threshold value for which two-stream instabilities onset. In section 6.4.1, we start with a model of the plasma dielectric function that assumes ions are cold (a fluid plasma dielectric function). This assumption is motivated by the experimental parameters in which ions are approximately room temperature (≈ 0.02 eV), while electrons are much hotter (≈ 0.7 eV). We will find that the cold ion model predicts instability whenever the flow speed of each ion species is different, because in this limit $v_{Ti} \to 0$, so $\Delta V_c \to 0$. Using $\Delta V = 0$ as one condition, and the Bohm criterion of equation 6.2 as the other, we find that each ion species leaves the plasma (and enters the sheath) at a common sound speed given by the system sound speed: $V_1 = V_2 = c_s$. This result is in agreement with previous experiments conducted in cold ion temperature plasmas [48–52, 100–102].

Although the cold ion result agrees with the previous experiments, all plasmas have finite ion temperatures and it is important to know how this might change the common sound speed result. In section 6.4.2 we account for the finite ion temperatures using a kinetic dispersion relation instead of the cold ion (fluid) approximation in order to calculate ΔV_c. We find that ΔV_c depends not only on the ion thermal speeds, but also on the density ratio of the two ion species (n_1/n_2). The predicted result is that ΔV_c is much smaller when n_1/n_2 is close to 1 than it is when n_1/n_2 is very large, or small. In section 6.5, we apply the more general condition $\Delta V = \Delta V_c$ to determine the Bohm criterion from equation 6.2. The prediction that the ion flow speeds at the sheath edge depend on their relative densities is a new result

that provides a convenient way to test our theory experimentally. In section 6.5, we show LIF data by Yip *et* al [54] that has already carried out this test. The data appears to confirm our predictions.

6.4.1 Cold Ion Model for Two-Stream Instabilities

Since we assume that the distribution functions of both ion species and electrons are Maxwellian, the dielectric function of equation 2.18 reduces to (see section 4.3)

$$\hat{\varepsilon}(\mathbf{k},\omega) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2 v_{Ts}^2} Z'\left(\frac{\omega - \mathbf{k}\cdot\mathbf{V}_s}{k v_{Ts}}\right), \tag{6.41}$$

in which Z is the plasma dispersion function and the derivative is with respect to the argument of Z. We are considering flowing Maxwellian ions, and stationary Maxwellian electrons. Typically, for ion waves, one assumes $(\omega - \mathbf{k}\cdot\mathbf{V})/kv_{Ti} \gg 1$ for ions and $\omega/kv_{Te} \ll 1$ for electrons. This is because the wave phase speed is typically on the order of the ion sound speed and electrons are much hotter than ions. We will see later that this approximation is not valid for capturing the two-stream instability when the relative ion flow $\Delta V = V_1 - V_2$ is on the order of the ion thermal speeds. However, we proceed to calculate the collisional friction using this ordering, and will consider how small ΔV can be accounted for in section 6.4.3. Applying these assumptions yields the fluid plasma dielectric function

$$\hat{\varepsilon}(\mathbf{k},\omega) = 1 - \frac{\omega_{p1}^2}{(\omega - \mathbf{k}\cdot\mathbf{V}_1)^2} - \frac{\omega_{p2}^2}{(\omega - \mathbf{k}\cdot\mathbf{V}_2)^2} + \frac{1}{k^2\lambda_{De}^2}. \tag{6.42}$$

The electron and ion Landau damping terms are both small in this limit.

Solving for the roots of equation 6.42, in order to determine the dispersion relation of the unstable modes, requires solving a quartic equation. Quartic equations can be solved analytically using Ferrari's method. In appendix D, we exactly solve for all four roots of equation 6.42 analytically. Two of these four solutions are stable ion sound waves (with $\omega \approx kc_s$), the other two are either damped or growing ion waves [with $\omega \approx k(V_1 + V_2)/2$] one of which can be unstable. However, the exact result for each mode is a very complicated equation that is essentially unusable for analytically evaluating $\mathbf{R}_{\mathrm{IE}}^{s-s}$. What we need is a simple approximation that can capture the dispersion relation of the one unstable root of equation 6.42 that we are interested in. The stable, or damped, waves do not contribute to enhancing the ion scattering, and thus we are not interested in them.

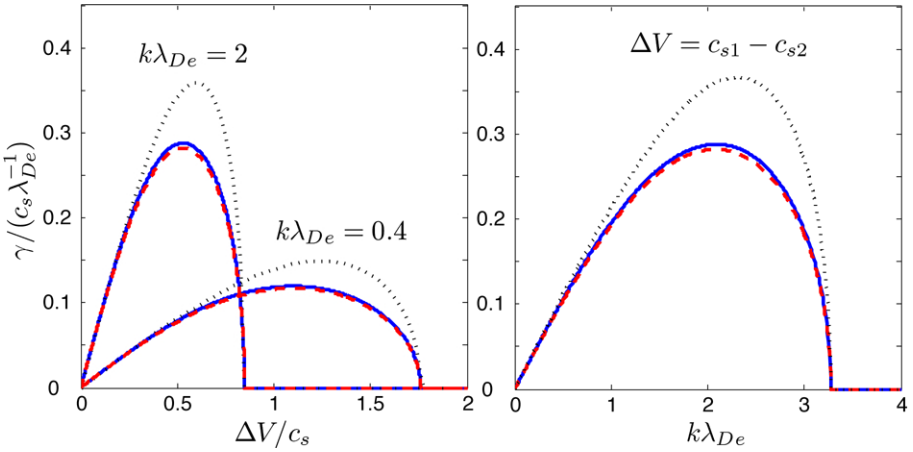

Figure 6.3: Normalized growth rates calculated for the parameters of [50] from a numerical solution of equation 6.42 (solid blue line), from the quadratic approximation of equation 6.47 (dashed red line) and from the approximation of equation 6.50 (dotted black line).

If we apply the substitution

$$\omega = \frac{1}{2}\mathbf{k}\cdot(\mathbf{V}_1 + \mathbf{V}_2) + \mathbf{k}\cdot\Delta\mathbf{V}\Omega \tag{6.43}$$

to equation 6.42, then the roots of equation 6.42 can be identified from the four solutions of the reduced quartic equation

$$\Omega^4 - \Omega^2\left(\frac{1}{2} + a\right) - \Omega\, ab + \frac{1}{16} - \frac{a}{4} = 0 \tag{6.44}$$

in which we have defined

$$a \equiv \frac{k^2 c_s^2}{(\mathbf{k}\cdot\Delta\mathbf{V})^2(1 + k^2\lambda_{De}^2)} \tag{6.45}$$

and

$$b \equiv \frac{\omega_{p1}^2 - \omega_{p2}^2}{\omega_{p1}^2 + \omega_{p2}^2}. \tag{6.46}$$

However, we are only interested in the one unstable solution. We will find that for this root $\Omega \sim b$ and $b < 1$ (for the sample plasma parameters $b \approx 1/2$) so the Ω^4 term can be neglected in equation 6.44, for finding the potentially unstable root of interest. The resulting quadratic equation yields the solutions

$$\Omega = -\frac{ab \pm \sqrt{a^2b^2 + (1/2 + a)(1/4 - a)}}{1 + 2a}. \tag{6.47}$$

Figure 6.3 show that equation 6.47 provides a very accurate approximation of the unstable root of the fluid plasma dielectric function from equation 6.42. However, equation 6.47 is still a bit complicated,

and we seek a further simplified form that can be used to analytically approximate $\mathbf{R}_{\rm IE}$. Noticing that $a > 1$ when $k\lambda_{De} < \sqrt{c_s^2/\Delta V^2 - 1}$, we can treat a as a large number for this part of k-space. Since $\Delta V \leq c_{s1} - c_{s2}$ in the presheath (even in the absence of friction), this is valid for at least $k\lambda_{De} \lesssim 1$ using the sample plasma parameters. In this limit, the leading term of equation 6.47 is $\Omega \approx -b/2 \pm i\sqrt{\alpha}/(1+\alpha)$ which is unstable for all k in the range of validity. Here we have defined

$$\alpha = \frac{n_1}{n_2}\frac{M_2}{M_1}. \tag{6.48}$$

When a becomes smaller than some critical value $a \leq a_c$, stabilization occurs and we account for this stabilization by using the approximation $\Omega \approx -b/2 \pm i\sqrt{\alpha(1 - a_c/a)}/(1+\alpha)$, in which a_c is obtained from equation 6.47. This gives $1/a_c = 1 + \sqrt{9 - 8b^2}$. With these, we arrive at an approximate dispersion relation for the unstable root: $\omega = \omega_R + i\gamma$, in which

$$\omega_R \approx \mathbf{k} \cdot \left(\frac{n_2}{n_e}\frac{c_{s2}^2}{c_s^2}\mathbf{V}_1 + \frac{n_1}{n_e}\frac{c_{s1}^2}{c_s^2}\mathbf{V}_2 \right) \tag{6.49}$$

is the real part, and

$$\gamma \approx \frac{k_\parallel \Delta V \sqrt{\alpha}}{1+\alpha}\sqrt{1 - \frac{k_\parallel^2 \Delta V^2}{k^2 \Delta V_{\rm up}^2}(1 + k^2\lambda_{De}^2)} \tag{6.50}$$

is an expression for the growth rate. The $\parallel$ direction is along $\Delta\mathbf{V}$ and

$$\Delta V_{\rm up}^2 \equiv c_s^2\left[1 + \sqrt{1 + 32\alpha/(1+\alpha)^2}\right] \tag{6.51}$$

is an upper limit above which the mode stabilizes.

Figure 6.3 shows that equation 6.50 can overestimate the growth rate by as much as 30%. However, we will show in section 6.4.2 that this quantitative difference will not affect our central conclusion. Applying equation 6.50 to calculate $\mathbf{R}_{\rm IE}$ will lead to underestimating the minimum distance ($z_{\rm min}$) that waves must grow before $\mathbf{R}_{\rm IE}$ dominates by up to 30%. Correcting for this error will be important when checking that $z_{\rm min}$ is much shorter than the presheath scale length l. We will find that $z_{\rm min}/l \sim 10^{-2}$, so a 30% correction to $z_{\rm min}$ is irrelevant to this discussion. Nevertheless, the 30% error can easily be tracked through the calculation and accounted for.

6.4.2 Calculation of Instability-Enhanced Collisional Friction

Next, we calculate the instability-enhanced collisional friction that results when the two-stream instability of section 6.4.1 is present. Recall from equation 6.14 that the instability-enhanced contribution

to the ion-ion collisional friction force density, assuming that both ion species are Maxwellian, is

$$\mathbf{R}_{\rm IE}^{1-2} = -\frac{m_1}{T_1}\int d^3v \int d^3v'\, f_1(\mathbf{v})\, f_2(\mathbf{v}')\, \mathcal{Q}_{\rm IE}^{1-2}\cdot\Delta\mathbf{V}. \tag{6.52}$$

Here we chosen the labels $s = 1$ and $s' = 2$. From equation 2.44, the instability-enhanced collisional kernel is

$$\mathcal{Q}_{\rm IE}^{1-2} = \frac{2q_2^2q_2^2}{\pi m_1}\int d^3k \frac{\mathbf{kk}}{k^4}\frac{\gamma}{(\omega_R-\mathbf{k}\cdot\mathbf{v})^2+\gamma^2}\frac{1}{(\omega_R-\mathbf{k}\cdot\mathbf{v}')^2+\gamma^2}\frac{\exp\big(2\gamma t\big)}{\big|\partial\hat{\varepsilon}/\partial\omega\big|^2_{\omega_R}}. \tag{6.53}$$

Since $v_{T1}, v_{T2} \ll c_{s,1}, c_{s,2} \sim V_1, V_2$ we approximate the ion distributions with delta functions in velocity space

$$f_1(\mathbf{v}) \approx n_1\delta(\mathbf{v}-\mathbf{V}_1) = n_1\delta(v_x)\delta(v_y)\delta(v_z-V_1) \tag{6.54}$$

and

$$f_2(\mathbf{v}') \approx n_2\delta(\mathbf{v}'-\mathbf{V}_2) = n_2\delta(v_x')\delta(v_y')\delta(v_z'-V_2). \tag{6.55}$$

Putting these into equation 6.52, the collisional friction is

$$\begin{aligned}\mathbf{R}_{\rm IE}^{1-2} = -\frac{2q_1^2q_2^2}{\pi T_1}n_1n_2\int d^3k\frac{\mathbf{kk}\cdot\Delta\mathbf{V}}{k^4}\frac{\exp\big(2\gamma t\big)}{\big|\partial\hat{\varepsilon}/\partial\omega\big|^2_{\omega_R}}\times \\ \times\int d^3v\int d^3v'\frac{\gamma\,\delta(v_x)\,\delta(v_y)\,\delta(v_z-V_1)\,\delta(v_x')\,\delta(v_y')\,\delta(v_z'-V_2)}{[(\omega_R-\mathbf{k}\cdot\mathbf{v})^2+\gamma^2][(\omega_R-\mathbf{k}\cdot\mathbf{v}')^2+\gamma^2]}.\end{aligned} \tag{6.56}$$

Upon evaluating the velocity integrals, this reduces to

$$\mathbf{R}_{\rm IE}^{1-2} = -\frac{2q_1^2q_2^2}{\pi T_1}n_1n_2\Delta\mathbf{V}\cdot\int d^3k\frac{\mathbf{kk}}{k^4}\frac{\gamma\,\exp\big(2\gamma t\big)}{\big|\partial\hat{\varepsilon}/\partial\omega\big|^2_{\omega_R}[(\omega_R-\mathbf{k}\cdot\mathbf{V}_1)^2+\gamma^2][(\omega_R-\mathbf{k}\cdot\mathbf{V}_2)^2+\gamma^2]}. \tag{6.57}$$

Taking the derivative of $\hat{\varepsilon}$ from equation 6.42 with respect to ω yields

$$\frac{\partial\hat{\varepsilon}}{\partial\omega} = \frac{2\omega_{p1}^2}{(\omega-\mathbf{k}\cdot\mathbf{V}_1)^3}+\frac{2\omega_{p2}^2}{(\omega-\mathbf{k}\cdot\mathbf{V}_2)^3}. \tag{6.58}$$

The real part of the unstable wave frequency, from equation 6.49, can be written in the alternative form

$$\omega_R = \frac{1}{2}k_\parallel\big[V_1(1+\beta)+V_2(1-\beta)\big] \tag{6.59}$$

in which we have defined

$$\beta \equiv \frac{n_2}{n_e}\frac{c_{s2}^2}{c_s^2}-\frac{n_1}{n_e}\frac{c_{s1}^2}{c_s^2}. \tag{6.60}$$

With this identification, we find that

$$\omega_R-\mathbf{k}\cdot\mathbf{V}_1 = -\frac{1}{2}(1-\beta)\mathbf{k}\cdot\Delta\mathbf{V} \tag{6.61}$$

and

$$\omega_R - \mathbf{k}\cdot\mathbf{V}_2 = \frac{1}{2}(1+\beta)\mathbf{k}\cdot\Delta\mathbf{V}. \tag{6.62}$$

Putting these into equation 6.58, and squaring the result, yields

$$\left|\frac{\partial\hat{\varepsilon}}{\partial\omega}\right|^2_{\omega_R} = \frac{256}{(\mathbf{k}\cdot\Delta\mathbf{V})^6}\left[\frac{\omega_{p1}^2}{(\beta-1)^3} + \frac{\omega_{p2}^2}{(\beta+1)^3}\right]^2. \tag{6.63}$$

The group velocity of the unstable waves is then

$$\mathbf{v}_g = \frac{\partial\,\omega_R}{\partial\mathbf{k}} = \frac{1}{2}\big[\mathbf{V}_1(1+\beta) + \mathbf{V}_2(1-\beta)\big]. \tag{6.64}$$

Again, we use cylindrical polar coordinates $\mathbf{k} = k_\perp\cos\theta\,\hat{x} + k_\perp\sin\theta\,\hat{y} + k_\parallel\hat{z}$, which implies

$$\mathbf{k}\mathbf{k}\cdot\Delta\mathbf{V} = \big(k_\parallel k_\perp\cos\theta + k_\parallel k_\perp\sin\theta + k_\parallel^2\big)\Delta V\,\hat{z}. \tag{6.65}$$

Noticing that this is the only place that angular dependence shows up in the integral of equation 6.57 (since γ, ω_R and $|\partial\hat{\varepsilon}/\partial\omega|^2_{\omega_R}$ are only functions of $|\mathbf{k}|$ and $k_\parallel$) the terms with $\cos\theta$ and $\sin\theta$ will vanish upon integrating over θ. If we also apply our assumption that $\omega_R - \mathbf{k}\cdot\mathbf{V} \ll \gamma$, we then have

$$\mathbf{R}_{\mathrm{IE}}^{1-2} = -\frac{2q_1^2q_2^2}{\pi T_1}n_1n_2\Delta\mathbf{V}\int d^3k\frac{k_\parallel^2}{k^4}\frac{\gamma\,\exp\big(2\gamma t\big)}{\left|\partial\hat{\varepsilon}/\partial\omega\right|^2_{\omega_R}(\omega_R - \mathbf{k}\cdot\mathbf{V}_1)^2\,(\omega_R - \mathbf{k}\cdot\mathbf{V}_2)^2}. \tag{6.66}$$

We also define the variable

$$A \equiv \frac{\Delta V_{up}^2}{\Delta V^2} - 1, \tag{6.67}$$

with which the growth rate from equation 6.50 can be written

$$\gamma = \frac{k_\parallel\Delta V^2}{\Delta V_{\mathrm{up}}}\frac{\sqrt{\alpha}}{1+\alpha}\sqrt{A - k_\parallel^2\lambda_{De}^2}. \tag{6.68}$$

Applying the same approximation that we used in equation 4.69 to find the $2\gamma t$ term in the Langmuir's paradox problem, we find

$$2\gamma t \approx \frac{2\gamma z}{v_g} = W\,k_\parallel\lambda_{De}\sqrt{A - k_\parallel^2\lambda_{De}^2}, \tag{6.69}$$

where we have defined

$$W \equiv \frac{2\sqrt{\alpha}}{1+\alpha}\frac{\Delta V^2}{v_g\,\Delta V_{up}}\frac{z}{\lambda_{De}}. \tag{6.70}$$

From equation 6.64, we can write the group speed as

$$v_g = \frac{n_2}{n_e}\frac{c_{s2}^2}{c_s^2}V_1 + \frac{n_1}{n_e}\frac{c_{s1}^2}{c_s^2}V_2. \tag{6.71}$$

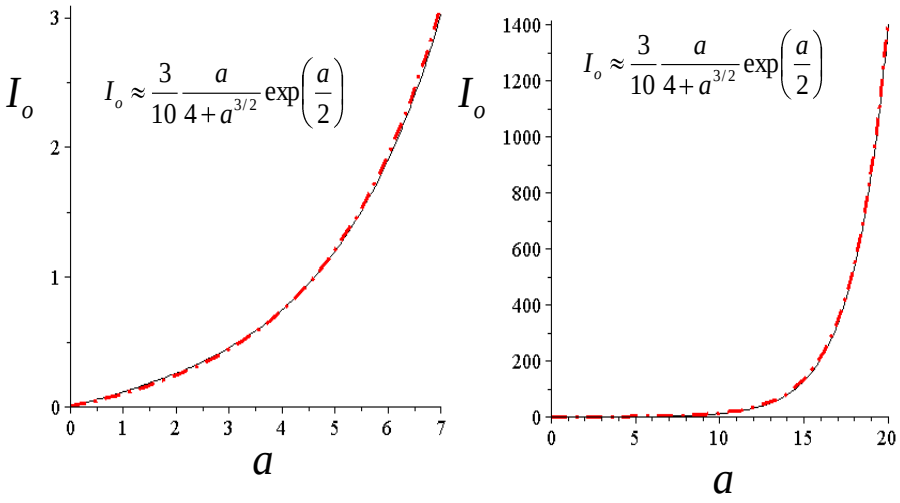

Figure 6.4: The approximation of the integral I_o, from equation 6.79 (red dashed line) accurately represents the actual result (black line).

Putting equations 6.61, 6.62, 6.63, 6.68 and 6.69, into equation 6.66, and evaluating the trivial θ integral, yields

$$\mathbf{R}_{\mathrm{IE}}^{1-2} = -\frac{q_1^2 q_2^2 n_1 n_2}{8T_1 \Delta V_{\mathrm{up}}} \frac{\sqrt{\alpha}}{1+\alpha} \frac{(1-\beta^2)^4\, \Delta V^5\, \hat{z}}{[\omega_{p1}^2(\beta+1)^3 + \omega_{p2}^2(\beta-1)^3]^2} \\ \times \int dk_\parallel\, k_\parallel^5 \sqrt{A - k_\parallel^2 \lambda_{De}^2} \exp\left(W k_\parallel \lambda_{De} \sqrt{A - k_\parallel^2 \lambda_{De}^2}\right) \int_0^\infty dk_\perp\, \frac{k_\perp}{k^4}. \tag{6.72}$$

The $k_\perp$ integral can be integrated analytically, which gives

$$\int_0^\infty dk_\perp \frac{k_\perp}{(k_\parallel^2 + k_\perp^2)^2} = \frac{1}{2k_\parallel^2}. \tag{6.73}$$

If we also identify the reference collision frequency from the stable plasma collisional friction in equation 6.37, equation 6.72 becomes

$$\mathbf{R}_{\mathrm{IE}}^{1-2} = -\frac{n_1 m_1 \nu_s}{64\sqrt{\pi}\ln\Lambda} \frac{\bar{v}_T}{\Delta V_{\mathrm{up}}} \frac{\sqrt{\alpha}}{1+\alpha} \frac{(1-\beta^2)^4\, \Delta V^5 \hat{z}}{[\omega_{p1}^2(\beta+1)^3 + \omega_{p2}^2(\beta-1)^3]^2}\, I, \tag{6.74}$$

in which the $k_\parallel$ integral is

$$I = \int_{-k_c}^{k_c} dk_\parallel\, k_\parallel^3 \sqrt{A - k_\parallel^2 \lambda_{De}^2} \exp\left(W\, k_\parallel \lambda_{De} \sqrt{A - k_\parallel^2 \lambda_{De}^2}\right). \tag{6.75}$$

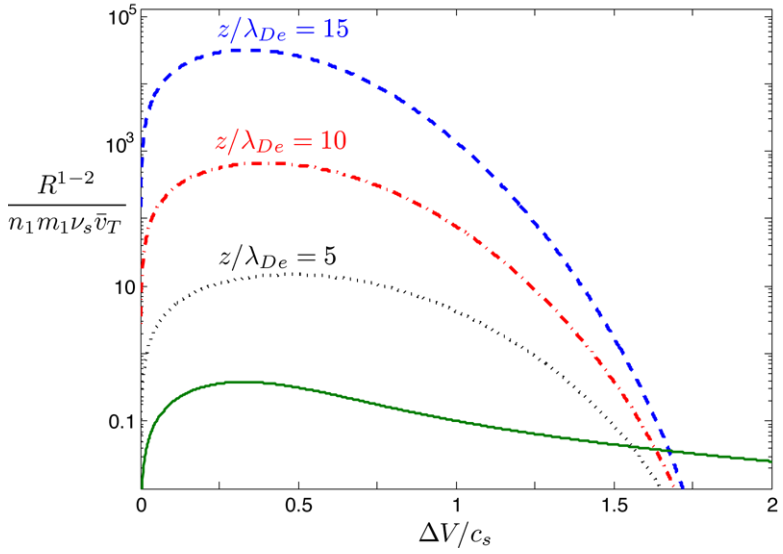

Figure 6.5: Normalized collisional friction force density for the parameters of the experiment in reference [50] due to Coulomb interactions in a stable plasma, calculated using equation 6.36, (solid green line) and due to instability-enhanced collective interactions from two stream instabilities, calculated using equation 6.80, for wave growth over a distances of $z/\lambda_{De} = 5$, 10 and 15 (dotted black line, dash-dotted red line, and dashed blue line).

The integration limit has been imposed simply to restrict the integration domain to unstable $k_\parallel$: $k_c = \sqrt{A}/\lambda_{De}$. The $k_\parallel$ outside of this domain rapidly damp and provide no contribution to this integral.

Using the substitution $x \equiv k_\parallel \lambda_{De}/\sqrt{A}$, the integral I becomes

$$I = \frac{A^{5/2}}{\lambda_{De}^4} \int_{-1}^{1} dx\, x^3 \sqrt{1-x^2}\, \exp\left(W A\, x \sqrt{1-x^2}\right). \tag{6.76}$$

This integral can be approximated by taking both the small argument expansion of the exponential and the asymptotic limit of the integral, then matching the results with a Padé approximation. For small $a = AW$, the small argument expansion yields

$$\int_{-1}^{1} dx\, x^3 \sqrt{1-x^2}\, \exp\left(a\, x \sqrt{1-x^2}\right) = \underbrace{\int_{-1}^{1} dx\, x^3 \sqrt{1-x^2}}_{=0} + a \underbrace{\int_{-1}^{1} dx\, x^4 (1-x^2)}_{=4/35} + \ldots \approx \frac{4}{35} a. \tag{6.77}$$

The asymptotic behavior of this integral for large a is

$$\int_{-1}^{1} dx\, x^3 \sqrt{1-x^2}\, \exp\left(a\, x \sqrt{1-x^2}\right) \approx \frac{3}{10} \frac{1}{\sqrt{a}} \exp\left(\frac{a}{2}\right). \tag{6.78}$$

It is hard to match the 4/35 and 3/10 numbers precisely, but a choice that works well for all a is

$$I_o \equiv \int_{-1}^{1} dx\, x^3 \sqrt{1-x^2}\, \exp\left(a\, x \sqrt{1-x^2}\right) \approx \frac{3}{10} \frac{a}{4+a^{3/2}} \exp\left(\frac{a}{2}\right). \tag{6.79}$$

Figure 6.4 show that equation 6.79 provides an excellent approximation for this integral over a very broad range of $a = AW$. We will only be interested in $2 \lesssim a \lesssim 10$ here.

Putting the results of equations 6.79 and 6.76 into equation 6.74, we find that the instability-enhanced collisional friction is

$$\mathbf{R}_{\mathrm{IE}}^{1-2} \approx -n_1 m_1 \nu_{12} \exp\left(W \frac{A}{2}\right) \Delta \mathbf{V} \tag{6.80}$$

in which we have defined the frequency

$$\nu_{12} \equiv \frac{\nu_s}{\ln \Lambda} \frac{3}{160\sqrt{\pi}} \frac{\bar{v}_T\, \Delta V^4}{\Delta V_{\mathrm{up}} c_s^4} \frac{A^{5/2} a}{4+a^{3/2}} \frac{\alpha^{5/2}(1+\alpha^{1/3})^2}{\alpha^2 - 1} \tag{6.81}$$

Figure 6.5 plots the instability-enhanced collisional friction from equation 6.80 for wave growth over distances of $z/\lambda_{De} = 5$, 10 and 15. For the plot, we have used $v_g \approx c_s$. Also shown is the stable plasma contribution to this friction using $\mathbf{R}_{\mathrm{LB}}^{1-2}$ from equation 6.36. Recall from the end of section 6.3 that the stable plasma contribution to the friction force density was about 10 times smaller than the other terms of the momentum balance equation, and thus it was neglected in previous theoretical work [43–47]. For instability-enhanced friction to be important requires $\mathbf{R}_{\mathrm{IE}}^{1-2} / \mathbf{R}_{\mathrm{LB}}^{1-2} \gtrsim 10$. We see from figure 6.5 that after growing only 15 Debye lengths, the two-stream instabilities have enhanced the collisional friction over 10^4 times the stable plasma level. The presehath length scale for this plasma is $l \approx 5$ cm and $\lambda_{De} \approx 6 \times 10^{-3}$ cm [50], so the wave growth distances shown in figure 6.5 are much shorter than the presheath length $z/l \approx 10^{-2}$.

Since a tenfold enhancement of $\mathbf{R}$ over the stable plasma level is required for instability-enhanced friction to become important, and for $z/\lambda_{De} = 15$ the enhancement is over 10^4, the distance that unstable waves must grow before instability-enhanced friction dominates the momentum balance equation is much shorter than the presheath length scale [even accounting for the $\lesssim 30\%$ error introduced by the approximation of γ from equation 6.50]. This shows that in the cold ion limit, the collisional friction between ion species is so strong that each species should continually have approximately the same speed throughout the presheath, and in particular at the sheath edge. Thus, the only solution to equation 6.2

is that each species obtain the system sound speed c_s at the sheath edge, which is consistent with the previous experimental literature [48–52, 100–102].

6.4.3 Accounting for Finite Ion Temperatures

In this section, we consider the effects of finite ion temperatures and show that they, as well as the density ratio of the ion species, can cause stabilization of the two-stream instabilities and change the common sound speed solution obtained in the last section. One simple way to show that stabilization occurs for $\Delta V \sim \mathcal{O}(v_{Ti})$, is to use the fluid plasma dielectric function with thermal corrections [33]

$$\hat{\varepsilon}(\mathbf{k},\omega) = 1 + \frac{1}{k^2\lambda_{De}^2} - \frac{\omega_{p1}^2}{(\omega - \mathbf{k}\cdot\mathbf{V}_1)^2 - v_{T1}^2/2} - \frac{\omega_{p2}^2}{(\omega - \mathbf{k}\cdot\mathbf{V}_2)^2 - v_{T2}^2/2}. \tag{6.82}$$

Repeating the procedure of section 6.4.1 to find the growth rate from this dielectric function, one finds that the lowest order contribution for $\Delta V \ll c_s$ is [17]

$$\gamma = \frac{\sqrt{\alpha}}{1+\alpha}\sqrt{k_\parallel^2\Delta V^2 - k^2\Delta V_c^2} \tag{6.83}$$

which is the growth rate if $\Delta V > \Delta V_c k/k_\parallel$ where

$$\Delta V_c \equiv \sqrt{\frac{1+\alpha}{2\alpha}}\sqrt{v_{T1}^2 + \alpha\, v_{T2}^2} \tag{6.84}$$

is the critical difference in ion flow speeds for instability to onset. Recall that $\alpha \equiv n_1 M_2/(n_2 M_1)$.

Equation 6.84 shows that the critical relative flow speed is $\mathcal{O}(v_{Ti})$. It also shows that there is a density ratio, as well as temperature dependence, on the critical relative flow speed for instability. The problem with equation 6.84 is that it is based on the fluid plasma dielectric function which is not valid for $\Delta V \sim \mathcal{O}(v_{Ti})$. The fluid plasma dielectric function assumes that $\omega - \mathbf{k}\cdot\mathbf{V}_i \gg v_{Ti}$, but we showed in equations 6.61 and 6.62 that $\omega - \mathbf{k}\cdot\mathbf{V}_i \propto \Delta V$, for both $i = 1, 2$; thus, the fluid approximation breaks down for $\Delta V \sim \Delta V_c$. Equation 6.84 can provide an order-of-magnitude estimate of the relative flow speeds, but a kinetic dielectric function must be used for a more accurate quantitative determination. We will develop such a model in this section.

Since we assume Maxwellian ion distribution functions, and are looking for ion waves that have a phase speed close to the ion sound speed, $\omega/kv_{Te} \ll 1$ and the kinetic dielectric function has the form

$$\hat{\varepsilon}(\mathbf{k},\omega) = 1 + \frac{1}{k^2\lambda_{De}^2} - \frac{\omega_{p1}^2}{k^2 v_{T1}^2} Z'\left(\frac{\omega - \mathbf{k}\cdot\mathbf{V}_1}{kv_{T1}}\right) - \frac{\omega_{p2}^2}{k^2 v_{T2}^2} Z'\left(\frac{\omega - \mathbf{k}\cdot\mathbf{V}_2}{kv_{T2}}\right). \tag{6.85}$$

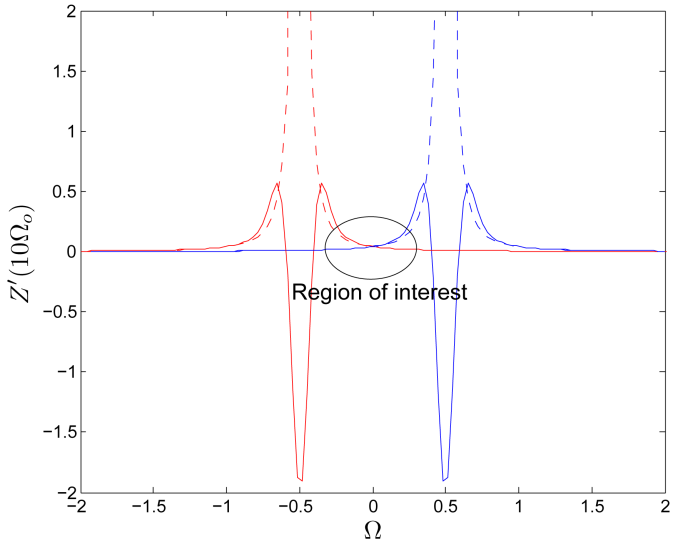

Figure 6.6: Plots of $Z'(10\Omega_o)$ for $\Omega_o = \Omega - 1/2$ (blue) and $\Omega_o = \Omega + 1/2$ (red). The 10 here is characteristic of a large ΔV since it is the coefficient representing $k_\parallel \Delta V/(kv_{Ti})$ (for each ion species). The solid lines show the exact Z' functions, and the dashed lines show the cold ion asymptotic approximation. In this case, the cold-ion approximation is good near the location where the unstable roots are found. When the 10 is replaced by a number of order unity (meaning $\Delta V \sim v_{Ti}$), the two Z' functions essentially overlap, and the cold-ion approximation fails.

As in section 6.4.1, we again apply the substitution

$$\omega = \frac{1}{2}\mathbf{k}\cdot\big(\mathbf{V}_1 + \mathbf{V}_2\big) + \mathbf{k}\cdot\Delta\mathbf{V}\Omega, \tag{6.86}$$

which show that $\omega - \mathbf{k}\cdot\mathbf{V}_1 = \mathbf{k}\cdot\Delta\mathbf{V}\big(\Omega - 1/2\big)$ and $\omega - \mathbf{k}\cdot\mathbf{V}_2 = \mathbf{k}\cdot\Delta\mathbf{V}\big(\Omega + 1/2\big)$. Putting this substitution into the dielectric function gives

$$\hat{\varepsilon}(\mathbf{k},\omega) = 1 + \frac{1}{k^2\lambda_{De}^2} - \frac{\omega_{p1}^2}{k^2 v_{T1}^2} Z'\left(\frac{\mathbf{k}\cdot\Delta\mathbf{V}\big(\Omega - 1/2\big)}{kv_{T1}}\right) - \frac{\omega_{p2}^2}{k^2 v_{T2}^2} Z'\left(\frac{\mathbf{k}\cdot\Delta\mathbf{V}\big(\Omega + 1/2\big)}{kv_{T2}}\right). \tag{6.87}$$

To find the dispersion relation, we set $\hat{\varepsilon} = 0$, which yields

$$1 + k^2\lambda_{De}^2 = \frac{n_1}{n_e}\frac{c_{s1}^2}{v_{T1}^2} Z'\left(\frac{\mathbf{k}\cdot\Delta\mathbf{V}\big(\Omega - 1/2\big)}{kv_{T1}}\right) + \frac{n_2}{n_e}\frac{c_{s2}^2}{v_{T2}^2} Z'\left(\frac{\mathbf{k}\cdot\Delta\mathbf{V}\big(\Omega + 1/2\big)}{kv_{T2}}\right). \tag{6.88}$$

Recall that for $w \ll 1$:

$$Z'(w) = -2i\sqrt{\pi}we^{-w^2} - 2 + 4w^2 - \frac{8}{3}w^4 + \ldots \tag{6.89}$$

and for $w \gg 1$:

$$Z'(w) = -2i\sigma\sqrt{\pi}we^{-w^2} + \frac{1}{w^2} + \frac{3}{2w^4} + \frac{15}{4w^6} + \ldots \tag{6.90}$$

The cold ion approximation from section 6.4.1 was based on the $|w| \gg 1$ asymptotic expansion for both ion species. Since $\omega - \mathbf{k}\cdot\mathbf{V}_i \propto \mathbf{k}\cdot\Delta\mathbf{V}$, we see from equation 6.88 that this is equivalent to assuming $\Delta V/kv_{Ti} \gg 1$. The Z' terms of equation 6.88 are plotted in figure 6.6 for a value $\Delta V/v_{Ti} = 10$, which is representative of the situation considered in section 6.4.1. Figure 6.6 shows that when $\Delta V/v_{Ti} \gg 1$, this asymptotic expansion does accurately model each of the Z' functions in the region where an unstable root can be found. The kinetic dielectric function reduces to a fluid model in this limit.

However, equation 6.88 also shows that if $\Delta V \sim \mathcal{O}(v_{Ti})$, neither the small argument expansion nor the asymptotic expansion is valid. This is a critical issue because we have seen that ΔV effectively cannot exceed ΔV_c because of instability-enhanced frictional forces, and from equation 6.84 we expect that $\Delta V_c \sim \mathcal{O}(v_{Ti})$. In this case, the Z' terms shown in figure 6.6 become much broader and overlap (as shown in figure 6.7). The asymptotic expansion that leads to the fluid theory used in section 6.4.1 cannot be used in this situation. Thus, we seek a new approximation of the plasma dispersion functions in equation 6.88 that can be used to determine ΔV_c directly from the kinetic theory.

Finding ΔV_c

For $\Delta V \approx \Delta V_c \sim \mathcal{O}(v_{Ti})$, the electron and vacuum terms of equation 6.88 are smaller than the ion terms by a factor of the ion to electron temperature ratio, which we assume to be large: $(1 + k^2\lambda_{De}^2) \ll T_e/T_i$. Thus, to find ΔV_c, we can neglect the vacuum and electron terms and solve

$$\frac{n_1}{n_2}\frac{T_2}{T_1}Z'\left(\frac{\mathbf{k}\cdot\Delta\mathbf{V}\big(\Omega - 1/2\big)}{kv_{T1}}\right) + Z'\left(\frac{\mathbf{k}\cdot\Delta\mathbf{V}\big(\Omega + 1/2\big)}{kv_{T2}}\right) = 0. \tag{6.91}$$

To resolve finite ion temperature effects, we consider what happens as ΔV is increased from zero. For very small ΔV, the Z' functions are very broad and the two terms of equation 6.91 essentially overlap. As ΔV increases from a very small value, the two Z' functions separate, as shown in figures 6.7 and 6.8. Unstable roots are found when the peaks of these two functions spread far enough apart. When this occurs, one can choose more appropriate points than the small or large argument from which to expand each of the Z' functions in a Taylor series. The appropriate choice of expansion points depends on the relative thermal speeds of the ions. We will chose to expand both functions about their positive peaks, which provides a good approximation when the ion thermal speeds are similar. We will still only be interested in the real part of the Z' expansion because the imaginary parts are small near these peaks.

We'd also expect any instabilities driven by the imaginary contribution (i.e., inverse Landau damping) to have a much smaller growth rate than the fluid instabilities.

Expanding $Z'(w)$ in a Taylor series about an arbitrary center point $w = c$ gives

$$\begin{aligned} Z'(w)\big|_{w=c} = &\left[-2 + 2c\sqrt{\pi}e^{-c^2}\mathrm{erfi}(c) - i2c\sqrt{\pi}e^{-c^2}\right] \\ &+ \left[4c - (4c^2-2)\sqrt{\pi}e^{-c^2}\mathrm{erfi}(c) + i(4c^2-2)\sqrt{\pi}e^{-c^2}\right](w-c) \\ &+ \left[4(1-c^2) + (4c^3-6c)\sqrt{\pi}e^{-c^2}\mathrm{erfi}(c) - i(4c^3-6c)\sqrt{\pi}e^{-c^2}\right](w-c)^2 + \mathcal{O}[(w-c)^3]. \end{aligned} \tag{6.92}$$

Recall that

$$\mathrm{erfi}(z) = -i\mathrm{erf}(iz) \tag{6.93}$$

is the "imaginary error function." We use it here because for a real center point c, $\mathrm{erfi}(c)$ is also a real number. We will keep terms up to quadratic order and use the notation

$$Z'(w) \approx a + b(w-c) + d(w-c)^2 \tag{6.94}$$

where

$$a \equiv -2 + 2c\sqrt{\pi}e^{-c^2}\mathrm{erfi}(c) - i2c\sqrt{\pi}e^{-c^2}, \tag{6.95}$$

$$b \equiv 4c - (4c^2-2)\sqrt{\pi}e^{-c^2}\mathrm{erfi}(c) + i(4c^2-2)\sqrt{\pi}e^{-c^2}, \tag{6.96}$$

and

$$d \equiv 4(1-c^2) + (4c^3-6c)\sqrt{\pi}e^{-c^2}\mathrm{erfi}(c) - i(4c^3-6c)\sqrt{\pi}e^{-c^2}. \tag{6.97}$$

If a center point c is specified, a, b and d are simply numbers that can be evaluated directly. After expanding each of the Z' functions about appropriate center points, equation 6.91 reduces to a quadratic equation that can be solved analytically. The trick is to pick the correct center points that are close to the location where the unstable mode is to be found. Choosing the appropriate c can be problem-dependent because the Z' function gets broader as the multiplier $\Delta V/v_{Ti}$ becomes smaller. So, for plasmas with very different ion thermal speeds, the $\Delta V/v_{Ti}$ is much bigger for one species than the other, which results in one Z' function being much broader than the other. For the experimental parameters that we are primarily interested in here, the ions are Ar+ and Xe+ with equal temperatures, so $v_{T1}/v_{T2} \approx \sqrt{M_2/M_1} = \sqrt{131/40} = 1.8$, which is not too far from 1. In this case, the Z' functions

have similar breadths and an appropriate center point, c, is at the positive peak of each function; as shown in figures 6.7 and 6.8.

We will proceed under the assumption that the ratio of ion thermal speeds is close to 1 (if it is in the range $1/4 \lesssim v_{T1}/v_{T2} \lesssim 4$ the following method should provide a reasonable estimate for ΔV_c). In this case the curves shown in figures 6.7 and 6.8 are representative of the Z' functions. An unstable root can arise in the region near the positive peaks of the Z' functions when these peaks separate with increasing ΔV. As the two peaks separate, the parabola that we use to model the sum of the two terms from equation 6.91 drops below the abscissa and predicts an unstable root. To capture the ΔV at which this occurs, we expand the real part of $Z'(w)$ about the peaks at $w = \pm 1.50201\ldots$. To within 0.1% this is $\pm 3/2$. Expanding $Z'(w)$ about $w = 3/2$ yields

$$\begin{aligned} Z'(w)\big|_{w=3/2} &= \left[-2 + 3\sqrt{\pi}\,\mathrm{erfi}(3/2)e^{-9/4} - i3\sqrt{\pi}e^{-9/4}\right] \\ &+ \left[6 - 7\sqrt{\pi}\,\mathrm{erfi}(3/2)e^{-9/4} + i7\sqrt{\pi}e^{-9/4}\right](w-3/2) \\ &+ \left[-5 + \frac{9}{2}\sqrt{\pi}\,\mathrm{erfi}(3/2)e^{-9/4} - i\frac{9}{2}\sqrt{\pi}e^{-9/4}\right](w-3/2)^2 + \mathcal{O}\left[(w-3/2)^3\right] \\ &= [0.57 - 0.56i] + [0.00 + 1.31i](w-3/2) + [-1.15 - 0.84i](w-3/2)^2 \\ &+ \mathcal{O}\left[(w-3/2)^3\right]. \end{aligned} \tag{6.98}$$

A second place that the peaks can separate is for negative Ω, see figure 6.8. Near this point, is appropriate to expand about $c = -3/2$, which gives

$$Z'(w)\big|_{w=-3/2} = [0.57+0.56i]+[0.00+1.31i](w+3/2)+[-1.15+0.84i](w+3/2)^2+\mathcal{O}([w+3/2]^3) \tag{6.99}$$

Thus we can take

$$Z'(w) \approx a + d(w-c)^2, \tag{6.100}$$

in which $a = 0.57$, $d = -1.15$ and $c = \pm 3/2$ to capture both of the possible locations for instability. The linear term is absent here because the real part of b is 0.00 – which is expected near the peak because Z' is flat there.

Putting these expansions into equation 6.91, we have

$$\frac{n_1}{n_2}\frac{T_2}{T_1}\left\{a + d\left[\frac{k_\parallel \Delta V}{k v_{T1}}\left(\Omega - 1/2\right) - c\right]^2\right\} + a + d\left[\frac{k_\parallel \Delta V}{k v_{T2}}\left(\Omega + 1/2\right) - c\right]^2 = 0. \tag{6.101}$$

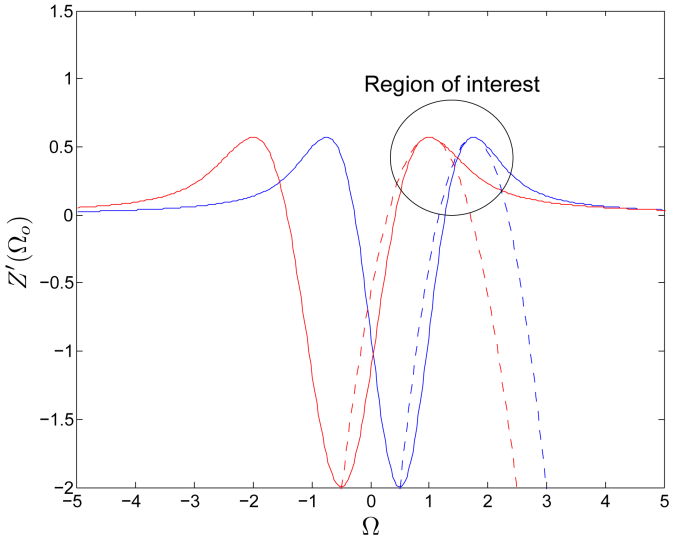

Figure 6.7: Plots of $Z'(1\Omega_o)$ for $\Omega_o = \Omega - 1/2$ and $\Omega_o = \Omega + 1/2$ (solid lines). The 1 here is characteristic of $\Delta V \sim v_{Ti}$. The dashed lines show the expansion about the peaks for $c = 3/2$.

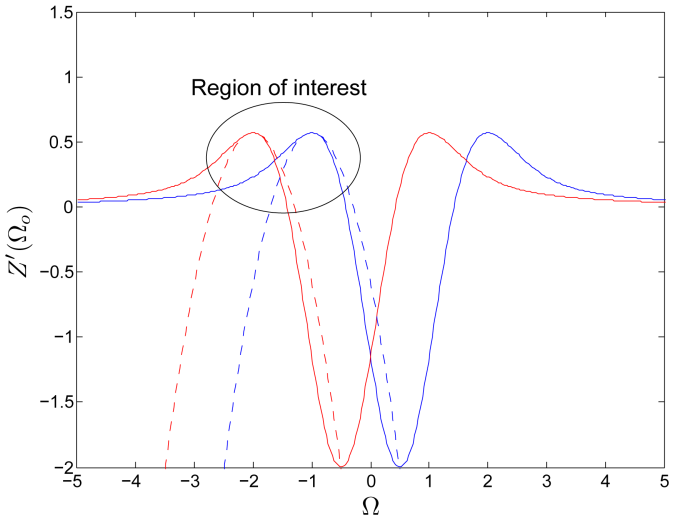

Figure 6.8: Plots of $Z'(1\Omega_o)$ for $\Omega_o = \Omega - 1/2$ and $\Omega_o = \Omega + 1/2$ (solid lines). The 1 here is characteristic of $\Delta V \sim v_{Ti}$. The dashed lines show the expansion about the peaks for $c = -3/2$.

Writing this in the common quadratic notation yields

$$\Omega^2\, d\left[\frac{n_1}{n_2}\frac{T_2}{T_1}\frac{k_\parallel^2\Delta V^2}{k^2v_{T1}^2}+\frac{k_\parallel^2\Delta V^2}{k^2v_{T2}^2}\right]-\Omega\, 2d\left[\frac{n_1}{n_2}\frac{T_2}{T_1}\frac{k_\parallel\Delta V}{kv_{T1}}\left(c+\frac{1}{2}\frac{k_\parallel\Delta V}{kv_{T1}}\right)+\frac{k_\parallel\Delta V}{kv_{T2}}\left(c-\frac{1}{2}\frac{k_\parallel\Delta V}{kv_{T2}}\right)\right] \quad (6.102)$$
$$+\,a\left(1+\frac{n_1}{n_2}\frac{T_2}{T_1}\right)+d\left[\frac{n_1}{n_2}\frac{T_2}{T_1}\left(c+\frac{1}{2}\frac{k_\parallel\Delta V}{kv_{T1}}\right)^2+\left(c-\frac{1}{2}\frac{k_\parallel\Delta V}{kv_{T2}}\right)^2\right]=0.$$

This quadratic has an unstable solution if

$$4d^2\left[\frac{n_1}{n_2}\frac{T_2}{T_1}\frac{k_\parallel\Delta V}{kv_{T1}}\left(c+\frac{1}{2}\frac{k_\parallel\Delta V}{kv_{T1}}\right)+\frac{k_\parallel\Delta V}{kv_{T2}}\left(c-\frac{1}{2}\frac{k_\parallel\Delta V}{kv_{T2}}\right)\right]^2-4d\left[\frac{n_1}{n_2}\frac{T_2}{T_1}\frac{k_\parallel^2\Delta V^2}{k^2v_{T1}^2}+\frac{k_\parallel^2\Delta V^2}{k^2v_{T2}^2}\right]\times \quad (6.103)$$
$$\times\left\{a\left(1+\frac{n_1}{n_2}\frac{T_2}{T_1}\right)+d\left[\frac{n_1}{n_2}\frac{T_2}{T_1}\left(c+\frac{1}{2}\frac{k_\parallel\Delta V}{kv_{T1}}\right)^2+\left(c-\frac{1}{2}\frac{k_\parallel\Delta V}{kv_{T2}}\right)^2\right]\right\}<0.$$

Simplifying this instability criterion gives

$$-4d^2\frac{n_1}{n_2}\frac{T_2}{T_1}\left[\frac{k_\parallel\Delta V}{kv_{T1}}\left(c-\frac{1}{2}\frac{k_\parallel\Delta V}{kv_{T2}}\right)-\frac{k_\parallel\Delta V}{kv_{T2}}\left(c+\frac{1}{2}\frac{k_\parallel\Delta V}{kv_{T1}}\right)\right]^2 \quad (6.104)$$
$$-\,4\,d\,a\left(\frac{n_1}{n_2}\frac{T_2}{T_1}\frac{k_\parallel^2\Delta V^2}{k^2v_{T1}^2}+\frac{k_\parallel^2\Delta V^2}{k^2v_{T2}^2}\right)\left(1+\frac{n_1}{n_2}\frac{T_2}{T_1}\right)<0.$$

Recall that $d<0$, so we will apply $d=-|d|$. Also, multiplying through by $k^4v_{T1}^2v_{T2}^2/(k_\parallel^2\Delta V^2)$ yields

$$-|d|\frac{n_1}{n_2}\frac{T_2}{T_1}\left[-k_\parallel\Delta V+c(kv_{T2}-kv_{T1})\right]^2+a\left(1+\frac{n_1}{n_2}\frac{T_2}{T_1}\right)\left(k^2v_{T1}^2+\frac{n_1}{n_2}\frac{T_2}{T_1}k^2v_{T2}^2\right)<0 \quad (6.105)$$

which is

$$\left|k_\parallel\Delta V+kc(v_{T1}-v_{T2})\right|>\sqrt{\frac{a}{|d|}\left(1+\frac{n_2}{n_1}\frac{T_1}{T_2}\right)\left(k^2v_{T1}^2+\frac{n_1}{n_2}\frac{T_2}{T_1}k^2v_{T2}^2\right)}. \quad (6.106)$$

Choosing to label species 1 and 2 so that $\Delta V>0$, we find that there is instability as long as

$$\Delta V>\frac{k}{k_\parallel}\Delta V_c \quad (6.107)$$

in which

$$\Delta V_c=c\,(v_{T2}-v_{T1})+\sqrt{\frac{a}{|d|}\left(1+\frac{n_2}{n_1}\frac{T_1}{T_2}\right)\left(v_{T1}^2+\frac{n_1}{n_2}\frac{T_2}{T_1}v_{T2}^2\right)}. \quad (6.108)$$

Recall that $a=0.57$, $|d|=1.15$ and $c=\pm1.5$. We are interested only in whichever unstable mode is excited first (when ΔV is increased from 0). For $v_{T1}>v_{T2}$, the $c=+3/2$ mode becomes unstable first (for the lowest ΔV), but for $v_{T2}>v_{T1}$, the $c=-3/2$ mode is unstable first. Thus, we can simply use $c(v_{T2}-v_{T1})\to-|c(v_{T2}-v_{T1})|$ to find the first unstable mode, regardless of which species is labeled 1

or 2. Putting in the relevant numbers ($a/|d| \approx 1/2$ to within 0.4%) gives

$$\boxed{\Delta V_c \approx -\frac{3}{2}|v_{T2} - v_{T1}| + \sqrt{\frac{1}{2}\left(1 + \frac{n_2}{n_1}\frac{T_1}{T_2}\right)\left(v_{T1}^2 + \frac{n_1}{n_2}\frac{T_2}{T_1}v_{T2}^2\right)}.} \tag{6.109}$$

Equation 6.109 provides a kinetic determination of the critical ΔV above which two-stream instabilities onset. It was derived based on the assumption that the ratio of ion thermal speeds is close to 1. In the next section, we show that equation 6.109 can be used to determine Bohm's criterion in plasmas with two ion species.

6.5 How Collisional Friction Can Determine the Bohm Criterion

In section 6.4.2 we found that when instability-enhanced friction onsets, the frictional force between ion species becomes so large that it forces the difference in their flow speeds back to the marginal value for instability onset. Because this system is so stiff, the critical speed for instability onset provides the following condition at the sheath edge

$$V_1 - V_2 = \Delta V_c. \tag{6.110}$$

If we take equality in the Bohm criterion from equation 6.2

$$\frac{n_1}{n_e}\frac{c_{s1}^2}{V_1^2} + \frac{n_2}{n_e}\frac{c_{s2}^2}{V_2^2} = 1, \tag{6.111}$$

this provides a second equation. Thus, with two equations we can solve for the two unknowns V_1 and V_2.

Putting equation 6.110 into the Bohm criterion of equation 6.111 yields

$$\frac{n_1}{n_e}\frac{c_{s1}^2}{V_1^2} + \frac{n_2}{n_e}\frac{c_{s2}^2}{(V_1 - \Delta V_c)^2} = 1. \tag{6.112}$$

This is a quartic equation to solve for V_1. Two of the solutions of this equation are imaginary and one is negative. We are only interested in the physically relevant positive real solution. From equation 6.109 we know that $\Delta V_c \sim \mathcal{O}(v_{Ti}) \ll V_1, V_2 \sim c_s$, so we expand equation 6.112 in a series for $\Delta V_c \ll V_1$, which yields

$$\frac{n_1}{n_e}\frac{c_{s1}^2}{V_1^2} + \frac{n_2}{n_e}\frac{c_{s2}^2}{V_1^2} + 2\frac{n_2}{n_e}\frac{c_{s2}^2\Delta V_c}{V_1^3} \approx 1. \tag{6.113}$$

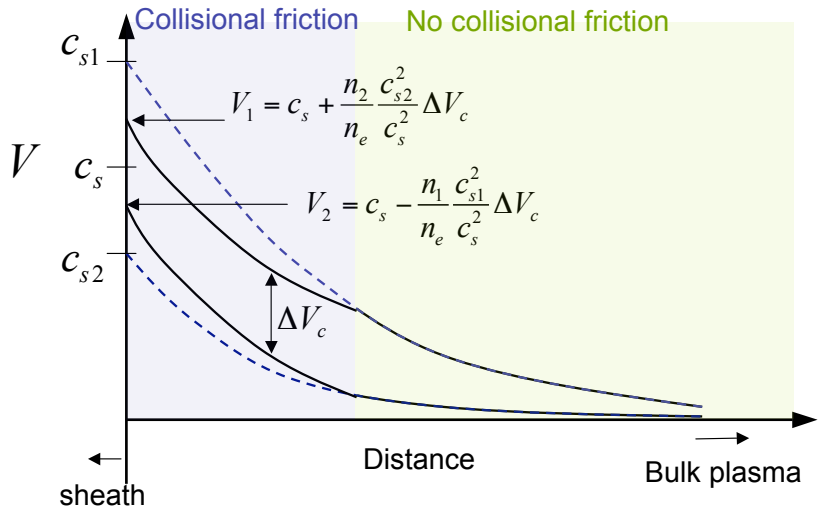

Figure 6.9: Sketch of predicted ion fluid speed profiles through the presheath. There is region with negligible ion-ion collisional friction near the plasma, and a region where it plays a dominant role near the boundary.

Identifying

$$c_s^2 \equiv \frac{n_1}{n_e}c_{s1}^2 + \frac{n_2}{n_e}c_{s2}^2, \tag{6.114}$$

equation 6.113 can also be written

$$\frac{c_s^2}{V_1^2} + 2\frac{n_2}{n_e}c_{s2}^2\frac{\Delta V_c}{V_1^3} = 1. \tag{6.115}$$

Multiplying by V_1^3 this is

$$V_1^3 = V_1 c_s^2 + 2\frac{n_2}{n_e}c_{s2}^2\Delta V_c. \tag{6.116}$$

Next, we apply the substitution

$$V_1 = c_s + \epsilon \tag{6.117}$$

and seek ϵ. Putting this into equation 6.116, we have

$$c_s^3 + 3c_s^2\epsilon + 3c_s\epsilon^2 + \epsilon^3 = c_s^3 + \epsilon c_s^2 + 2\frac{n_2}{n_e}c_{s2}^2\Delta V_c. \tag{6.118}$$

Since the c_s^3 terms cancel, we see that $\epsilon \sim \Delta V_c \sim \mathcal{O}(v_{Ti})$, which is small compared to c_s. Thus we can neglect the ϵ^2 and ϵ^3 terms compared to the ϵ terms. Doing this yields

$$\epsilon = \frac{n_2}{n_e}\frac{c_{s2}^2}{c_s^2}\Delta V_c. \tag{6.119}$$

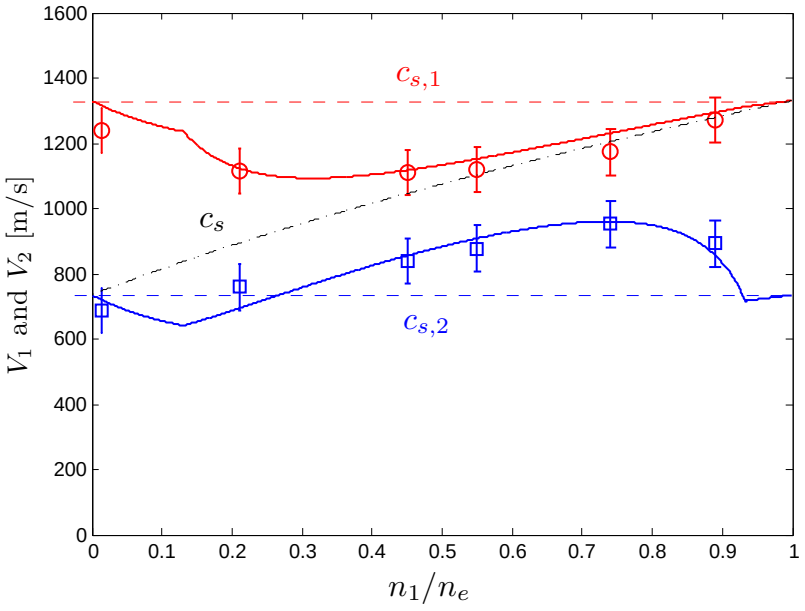

Figure 6.10: Plot of the ion flow speeds at the sheath edge from the theoretical prediction using equations 6.120 and 6.121 with ΔV_c from equation 6.109 (solid lines) and the experimental measurements of Yip, Hershkowitz and Severn [54]. Here the ion species are Ar+ (labeled 1 and shown in red) and Xe+ (labeled 2 and shown in blue). Also shown are the individual sound speeds (dashed red and blue lines) and the common system sound speed (dash-dotted black line).

Thus, we have

$$\boxed{V_1 \approx c_s + \frac{n_2}{n_e}\frac{c_{s2}^2}{c_s^2}\Delta V_c.} \tag{6.120}$$

Putting this into $V_1 - V_2 = \Delta V_c$ yields

$$\boxed{V_2 \approx c_s - \frac{n_1}{n_e}\frac{c_{s1}^2}{c_s^2}\Delta V_c.} \tag{6.121}$$

Equations 6.120 and 6.121 show that accounting for finite (but still small) ion temperatures leads to the result that the ion flow speed of each species at the sheath edge is close to the system sound speed c_s, but can differ from it by an amount that depends on the ion thermal speeds, as well as the density of each ion species. A schematic depiction of the presheath for this situation is shown in figure 6.9. This density dependence provides a convenient parameter that can be varied in experiments to test our theory. In fact, such an experiment has already been performed by Yip, Hershkowitz and Severn [54] (using LIF) to test equations 6.120 and 6.121, where ΔV_c is given by equation 6.109. The results of this

test are shown in figure 6.10. In this figure, we have used $\Delta V = \Delta V_c$ for $\Delta V_c < c_{s1} - c_{s2}$ (whenever instabilities are expected to be present) and $\Delta V = c_{s1} - c_{s2}$ when $\Delta V_c > c_{s1} - c_{s2}$ (whenever instabilities are not expected to be present). The relevant temperatures from the experiment where $T_1 \approx T_2 = 0.04$ eV and $T_e = 0.7$ eV. We have also labeled Ar+ ions as species 1 and Xe+ ions as species 2.

Figure 6.10 shows that the experimental data agree very well with our theory over a broad range of ion density ratios. The figure also shows the previous theoretical prediction of Franklin [43–47] that each ion species obtains its individual sound speed, and the solution proposed in previous experimental work [48–52, 100–102] that each ion species obtains the common system sound speed. The data does not seem to support either of these previous proposals over the whole range of ion concentrations. Franklin's solution agrees well with the data when the density ratio is either large, or small, in which case each ion species obtains a speed close to its individual sound speed. In this situation, our theory does not predict that any two-stream instabilities will be present, and our result converges to Franklin's. The common sound speed solution appears close when the ion density ratio is near 1 (in the plot, $n_1/n_e = 1/2$). In this case, our theory predicts instability-enhanced friction will be present for a small, but finite, difference in ion flow speeds. Our prediction that the speed at which each ion species falls into a sheath depends on the density of that species is not a feature of previous theories (or experiments). Figure 6.10 confirms that this is a qualitative feature of the physics. It also shows excellent quantitative agreement with the predictions of our theory.

Chapter 7

Conclusions

In this dissertation, a kinetic equation for collective interactions that accounts for electrostatic instabilities in unmagnetized plasmas was derived and applied to two unsolved problems in low-temperature plasma physics: Langmuir's paradox and determining the Bohm criterion for multiple-ion-species plasmas. Our theory generalizes the Lenard-Balescu kinetic equation to describe wave-particle scattering in weakly unstable plasmas, in addition to the particle-particle scattering from conventional Coulomb interactions that dominates in stable plasmas (the Lenard-Balescu equation assumes that the plasma is stable). We used two independent methods to arive at this equation: the dressed test particle approach in section 2.1 and the BBGKY hierarchy in section 2.2. An important feature of the resultant collision operator, equation 2.45, is that it can be written in the Landau form with both drag and diffusion terms. Another important feature is that the total collision operator consists of a sum of component species collision operators: $C(f_s) = \sum_{s'} C(f_s, f_{s'})$.

In chapter 3, we showed that the resultant collision operator obeys important physical properties such as conservation laws and the Boltzmann $\mathcal{H}$-theorem. Section 3.4.7 provided a proof that, within the weak-instability approximation $\gamma/\omega_R \ll 1$, instability-enhanced collisions shorten the timescale for which equilibration of individual species distribution functions occurs. The unique equilibrium for each species was shown to be a Maxwellian, even when wave-particle scattering from instabilities is the dominant scattering mechanism. Collisions within individual species cause equilibration on the fastest timescales. On longer timescales, the different species equilibrate with one another as well. On this long timescale, the unique thermodynamic equilibrium state is a Maxwellian plasma where each species has the same flow velocity and temperature. The most important of these properties in the applications we considered were that the unique equilibrium state from self-collisions ($s' = s$) is Maxwellian, and that momentum is conserved for collisions between individual species in the plasma.

The instability-enhanced contribution to the total collision operator was shown to have a diffusive form that fits the framework of conventional quasilinear theory, but for which the continuing source of fluctuations was self-consistently accounted for by its association with discrete particle motion. This led to a determination of the spectral energy density that is absent in conventional quasilinear theory. The feature that our theory self-consistently accounts for the origin of fluctuations due to discrete particles distinguishes this work from previous theories. Previous kinetic theories of weakly unstable plasmas, such as Friemann and Rutherford [5], or Rogister and Oberman [6], did not account for a fluctuation source. Thus, like quasilinear theory, using these theories requires an external determination of the source fluctuation spectrum. If the source is from applied waves, for example by an antenna, the source spectrum might be easy to determine. However, if the fluctuations are generated internal to the plasma itself, the spectrum can be difficult to determine. The advantage of our approach is that the source fluctuations are self-consistently accounted for as long as they arise internal to the plasma.

Our kinetic equation connects the work of Kent and Taylor [11], which introduced the concept that collective fluctuations arise from discrete particle motion, with previous kinetic and quasilinear theories for scattering in weakly unstable plasmas, such as Rogister and Oberman [6], which treated the fluctuations as independent of the discrete particle motions. Kent and Taylor [11] developed a theory which described the evolution of the amplitude of unstable waves from their origin as discrete particle fluctuations, and emphasized drift-wave instabilities in magnetized inhomogeneous systems. They did not develop a collision operator for particle scattering from the unstable waves. Baldwin and Callen [12] did derive a collision operator that accounted for the discrete-particle origin of fluctuations for the case of loss-cone instabilities in magnetic mirror machines. In this dissertation, we have considered a general formulation for unmagnetized plasmas. However, the basic result that the collision frequency due to instability-enhanced interactions scales as the product of $\delta/\ln\Lambda$ and the energy amplification due to fluctuations [$\sim \exp(2\gamma t)$] is common to this work and that of Baldwin and Callen [12]. Here δ is typically a small number $\delta \sim 10^{-2} - 10^{-3}$, which depends on the fraction of wave-number space that is unstable. Although our theory is limited by the assumption that the fluctuation amplitude be linear, we have found that instabilities can enhance the collision frequency by at least a few orders of magnitude before nonlinear wave amplitudes are reached.

In chapter 4, we applied our kinetic theory to the unsolved problem of Langmuir's paradox [13–15]. Langmuir's paradox is a measurement which showed anomalously fast electron scattering and equilibration to a Maxwellian in a low-temperature gas-discharge plasma. Langmuir's experiments were conducted in a 3 cm diameter mercury plasma, with glass walls, that was energized by electrons emitted from a hot filament (essentially a gas-filled incandescent light bulb). Langmuir measured the electron distribution function in this plasma with an electrostatic probe (now called a Langmuir probe) and found that it was Maxwellian to all diagnosable energies (which was greater than 50 eV). This was a surprising result because Langmuir knew that sheaths form near the plasma boundaries to reflect most of the electrons and maintain a quasineutral steady-state in the plasma. The sheath energy in Langmuir's plasma was approximately 10 eV. Since the electron-electron scattering length in his discharge was estimated to be approximately 30 cm, which was ten times the diameter of his plasma, Langmuir expected the electron distribution to be depleted for energies greater than the sheath energy since these electrons rapidly escape the plasma. Gabor named this anomaly "Langmuir's paradox" in 1955 [15], and this has remained a serious discrepancy in the kinetic theory of gas-discharge physics.

We focused on the plasma-boundary transition region of Langmuir's discharge, in particular the presheath. The presheath is a region where the ion flow speed transitions from essentially zero in the bulk plasma to the ion sound speed at the sheath edge. We showed that in plasmas where $T_e \gg T_i$, such as Langmuir's, that ion-acoustic instabilities are excited in the presheath. By applying our basic theory from chapter 2, we showed that the ion-acoustic instabilities could increase the electron-electron scattering frequency by more than two orders of magnitude. Furthermore, we could use the property from section 3.4.7 that the unique equilibrium distribution for these collisions is a Maxwellian to show that Langmuir's measurements could be explained by this instability-enhanced collision mechanism. The ion-acoustic instabilities in this problem convect toward the plasma boundaries and are lost before reaching a large enough amplitude that nonlinear wave-wave interactions become important; see section 4.4.2. Thus, our basic kinetic theory is well suited to describe this problem.

In chapter 5, we discussed basic aspects of another important plasma-boundary transition problem; the Bohm criterion. The criterion that Bohm first derived [38], assumed that ions in the plasma were monoenergetic and that electrons had a Maxwellian distribution. From this, Bohm showed that ions must be supersonic as they leave a plasma and enter a sheath, i.e., $V_i \geq c_s$. Since Bohm's time, this

criterion has been studied in much greater detail. One subset of these studies has aimed to find a kinetic version of the Bohm criterion. The theories that have been proposed in this area are based on taking a v^{-1} moment of the Vlasov equation. In section 5.1.3, we showed that this approach is not appropriate for most plasmas. It leads to a condition that has unphysical divergences, and also places unphysical importance on the part of the ion or electron distribution functions with low energy. The kinetic version of a Bohm criterion proposed in these previous works predicted a substantially different condition than the fluid predictions of $V_i \geq c_s$, if either the ion, or electron, distribution function had any contribution near zero velocity. In section 5.2, we developed an alternative kinetic Bohm criterion based on positive-velocity moments of the full kinetic equation. This result does not suffer from the unphysical divergences of the previous theories, or put undue emphasis on low energy particles. It essentially confirms the fluid predictions for most plasmas of interest. For example, we showed that in most low-temperature plasmas, ions are collisional in the presheath and one can expect that they have a Maxwellian distribution with flow. In this case, our criterion reduces to the conventional Bohm criterion $V_i \geq c_s$, where V_i is now the ion flow speed. For the same plasma, the previous kinetic Bohm criteria gives the condition $\infty \leq 1/T_e$, which is not a useful criterion (nor is it true).

If the plasma contains more than one species of ions, the Bohm criterion does not uniquely determine the speed of each ion species as it leaves the plasma. It provides a single condition, in as many unknowns as there is ion species in the plasma. In chapter 6, we considered the problem of how to determine the solution of the Bohm criterion when more than one species of positive ions is present. We focused on the case of two positive ion species. Previous theoretical and experimental work on this problem did not agree. For example, the theoretical work of Franklin [43–47] predicted that the speed of each ion species should be close to its individual species sound speed as it leaves the plasma: $V_i \approx c_{s,i} = \sqrt{T_e/M_i}$. Experimental work [48–52, 100–102], on the other hand, measured that the speed of each ion species was often much closer to a common speed at the sheath edge, given by the system sound speed: $V_i \approx c_s \equiv \sqrt{(n_i/n_e)c_{s,i}^2}$.

We showed that the reason for this discrepancy is ion-ion friction between the two ion species that is greatly enhanced by two-stream instabilities. As ions of different mass (or charge) are accelerated by the presheath electric field, their flow speeds separate. When the difference in their flow speeds exceeds a critical value of order the ion thermal speeds, $\Delta V > \Delta V_c = \mathcal{O}(v_{Ti})$, two stream instabilities arise.

Two stream instabilities are a virulent fluid instability that is convective. Using our kinetic theory from chapter 2, we showed that these instabilities generate a huge collisional friction force between the two ion species, within about 10 Debye lengths of the location where they become excited. This leads to a very stiff system whereby the difference in ion flow speeds cannot exceed the critical value at which they turn on. We showed in chapter 6 that this provides a second condition that ions must satisfy, $\Delta V = V_1 - V_2 = \Delta V_c$, which determines the Bohm criterion. Since ΔV_c depends on the relative densities of the ion species, this theory can easily be tested experimentally by varying the relative concentrations of the ion species and measuring the corresponding ion speeds at the sheath edge. We showed the results of such an experimental test in figure 6.10, which was conducted by Yip *et* al [54], and which agreed well with our predictions. In this application, our kinetic theory remaines valid because the instability-enhanced collisional friction modifies the plasma dielectric to limit the instability amplitude so that nonlinear fluctuation levels are never reached.

Appendix A

Rosenbluth Potentials

A.1 Definition of the Rosenbluth Potentials

In this appendix, we review properties of the Rosenbluth potentials and evaluate them explicitly for a Maxwellian distribution function (including flow). The H Rosenbluth potential is defined as [22]

$$H_{s'}(\mathbf{v}) \equiv \frac{m_s}{m_{ss'}} \int d^3v' \frac{f_{s'}(\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|} \tag{A.1}$$

and the G as

$$G_{s'}(\mathbf{v}) = \int d^3v'\, f_{s'}(\mathbf{v}')|\mathbf{v}-\mathbf{v}'|. \tag{A.2}$$

Some useful velocity-space derivative properties of the Rosenbluth potentials are

$$\frac{\partial^2}{\partial \mathbf{v}^2} H_{s'}(\mathbf{v}) = -4\pi \frac{m_s}{m_{ss'}} f_{s'}(\mathbf{v}), \tag{A.3}$$

$$\frac{\partial^2}{\partial \mathbf{v}^2} G_{s'}(\mathbf{v}) = 2\frac{m_{ss'}}{m_s} H_{s'}(\mathbf{v}), \tag{A.4}$$

and

$$\frac{\partial^2}{\partial \mathbf{v}^2}\frac{\partial^2}{\partial \mathbf{v}^2} G_{s'}(\mathbf{v}) = -8\pi f_{s'}(\mathbf{v}). \tag{A.5}$$

Some other useful identities when working with the Rosenbluth potentials are

$$\frac{\partial u}{\partial \mathbf{v}} = \frac{\mathbf{u}}{u} \quad , \quad \frac{\partial}{\partial \mathbf{v}}\frac{1}{u} = -\frac{\mathbf{u}}{u^3} \quad , \quad \frac{\partial^2 u}{\partial \mathbf{v}\partial \mathbf{v}} = \frac{u^2\mathcal{I} - \mathbf{uu}}{u^3}, \tag{A.6}$$

$$\frac{\partial^2}{\partial \mathbf{v}^2}\frac{1}{u} = \left(\frac{\partial}{\partial \mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{v}}\right)\frac{1}{u} = -4\pi\delta(\mathbf{u}) = -4\pi\delta(\mathbf{v}-\mathbf{v}'), \tag{A.7}$$

$$\frac{\partial^2 u}{\partial \mathbf{v}^2} = \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\partial u}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{\mathbf{u}}{u}\right) = \frac{2}{u} \tag{A.8}$$

and

$$\frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{u^2\mathcal{I} - \mathbf{uu}}{u^3}\right) = \left(\frac{\partial}{\partial \mathbf{v}}\frac{1}{u}\right) \cdot \mathcal{I} - \left(\frac{\partial}{\partial \mathbf{v}}\frac{1}{u^3}\right) \cdot \mathbf{uu} - \frac{1}{u^3}\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{uu} = -2\frac{\mathbf{u}}{u^3}. \tag{A.9}$$

A.2 Rosenbluth Potentials for a Flowing Maxwellian Background

When working with Maxwellians, it is convenient to use the Maxwell integral

$$\psi(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x dt\, \sqrt{t} e^{-t} \tag{A.10}$$

which satisfies the properties

$$\psi'(x) = \frac{d\psi}{dx} = \frac{2}{\sqrt{\pi}} \sqrt{x} e^{-x} \quad \text{and} \quad \psi + \psi' = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} dy\, e^{-y^2} = \text{erf}(\sqrt{x}). \tag{A.11}$$

First, we calculate $G_{s'}(\mathbf{v})$ using for a background (s' species) that has a flowing Maxwellian distribution

$$f_{M,s'}(\mathbf{v}') = \frac{n_{s'}}{\pi^{3/2} v_{Ts'}^3} \exp\left[-\frac{(\mathbf{v}' - \mathbf{V}_{s'})^2}{v_{Ts'}^2} \right]. \tag{A.12}$$

Putting this distribution into equation A.2, yields

$$G_{s'}(\mathbf{v}) = \frac{n_{s'}}{\pi^{3/2} v_{Ts'}^3} \int d^3u\, u \exp\left[-\frac{(\mathbf{v} - \mathbf{u} - \mathbf{V}_{s'})^2}{v_{Ts'}^2} \right], \tag{A.13}$$

in which $\mathbf{u} \equiv \mathbf{v} - \mathbf{v}'$. Redefining $\mathbf{u} \to -\mathbf{u}$ and defining

$$\mathbf{w}_{s'} \equiv \mathbf{v} - \mathbf{V}_{s'} \tag{A.14}$$

gives

$$G_{s'}(\mathbf{v}) = \frac{n_{s'}}{\pi^{3/2} v_{Ts'}^3} \int d^3u\, u \exp\left[-\frac{(\mathbf{u} + \mathbf{w}_{s'})^2}{v_{Ts'}^2} \right]. \tag{A.15}$$

Using spherical coordinates and aligning $\mathbf{w}_{s'}$ in the $\hat{z}$ direction, so $\mathbf{u} \cdot \mathbf{w}_{s'} = u w_{s'} \cos\theta$ gives

$$G_{s'}(\mathbf{v}) = \frac{n_{s'}}{\pi^{3/2} v_{Ts'}^3} \int_0^\infty du\, u^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta\, \sin\theta\, u \exp\left[-\frac{u^2 + w_{s'}^2 + 2u w_{s'} \cos\theta}{v_{Ts'}^2} \right]. \tag{A.16}$$

Doing the trivial ϕ integral and making the variable substitution $x = \cos\theta$ so $dx = -\sin\theta d\theta$, this is

$$\begin{aligned} G_{s'}(\mathbf{v}) &= \frac{2\pi n_{s'}}{\pi^{3/2} v_{Ts'}^3} \int_0^\infty du\, u^3 \exp\left(-\frac{u^2 + w_{s'}^2}{v_{Ts'}^2} \right) \int_{-1}^1 dx \exp\left(-\frac{2u w_{s'} x}{v_{Ts'}^2} \right) \\ &= -\frac{n_{s'}}{\sqrt{\pi} v_{Ts'} w_{s'}} \int_0^\infty du\, u^2 \left[e^{-(u + w_{s'})^2 / v_{Ts'}^2} - e^{-(u - w_{s'})^2 / v_{Ts'}^2} \right]. \end{aligned} \tag{A.17}$$

Substituting in $y = (u + w_{s'})/v_{Ts'}$ into the first integral and $y = -(u - w_{s'})/v_{Ts'}$ into the second gives

$$\begin{aligned} G_{s'}(\mathbf{v}) &= -\frac{n_{s'}}{\sqrt{\pi} v_{Ts'} w_{s'}} \left[\int_{w_{s'}/v_{Ts'}}^\infty dy\, v_{Ts'} \left(v_{Ts'}^2 y^2 - 2 w_{s'} v_{Ts'} y + w_{s'}^2 \right) e^{-y^2} \right. \\ &+ \left. \int_{w_{s'}/v_{Ts'}}^{-\infty} dy\, v_{Ts'} \left(v_{Ts'}^2 y^2 - 2 w_{s'} v_{Ts'} y + w_{s'}^2 \right) e^{-y^2} \right]. \end{aligned} \tag{A.18}$$

Using the substitution

$$x \equiv \frac{w_{s'}^2}{v_{Ts'}^2} \tag{A.19}$$

and the fact $\int_{\sqrt{x}}^{-\infty} = -\int_{-\infty}^{\sqrt{x}} = -\int_{-\infty}^{\infty} + \int_{\sqrt{x}}^{\infty}$ gives

$$G_{s'}(\mathbf{v}) = -\frac{n_{s'} v_{Ts'}}{\sqrt{\pi}\sqrt{x}}\left[-4\sqrt{x}\int_{\sqrt{x}}^{\infty} dy\, y e^{-y^2} + \int_{\sqrt{x}}^{\infty} dy\, e^{-y^2}\left(2y^2+2x\right) - \int_{-\infty}^{\infty} dy e^{-y^2}\left(y^2-2\sqrt{x}y+x\right)\right]. \tag{A.20}$$

The last $2\sqrt{x}y$ term vanishes due to odd symmetry. Also, noting that

$$\int_{-\infty}^{\infty} dy e^{-y^2}\left(y^2+x\right) = 2\int_{0}^{\infty} dy e^{-y^2}\left(y^2+x\right) \tag{A.21}$$

and for the middle term that $\int_{\sqrt{x}}^{\infty} = \int_0^{\infty} - \int_0^{\sqrt{x}}$, we find

$$G_{s'}(\mathbf{v}) = \frac{n_{s'} v_{Ts'}}{\sqrt{x}}\frac{2}{\sqrt{\pi}}\left[2\sqrt{x}\int_{\sqrt{x}}^{\infty} dy\, y e^{-y^2} + \int_0^{\sqrt{x}} dy\, y^2 e^{-y^2} + x\int_0^{\sqrt{x}} dy e^{-y^2}\right]. \tag{A.22}$$

Applying the definition for error functions

$$\int_{\sqrt{x}}^{\infty} dy\, y e^{-y^2} = \frac{1}{2}e^{-x} \quad , \quad \int_0^{\sqrt{x}} dy\, y^2 e^{-y^2} = -\frac{\sqrt{x}}{2}e^{-x} + \frac{\sqrt{\pi}}{4}\mathrm{erf}\left(\sqrt{x}\right) \quad \text{and} \quad \int_0^{\sqrt{x}} dy\, e^{-y^2} = \frac{\sqrt{\pi}}{2}\mathrm{erf}\left(\sqrt{x}\right) \tag{A.23}$$

we find

$$G_{s'}(\mathbf{v}) = n_{s'} v_{Ts'}\frac{1}{\sqrt{x}}\left[(x+\frac{1}{2})\mathrm{erf}\left(\sqrt{x}\right) + \frac{\sqrt{x}}{\sqrt{\pi}}e^{-x}\right], \tag{A.24}$$

which in terms of Maxwell integrals is

$$\boxed{G_{s'}(\mathbf{v}) = n_{s'} v_{Ts'}\frac{1}{\sqrt{x}}\left[(1+x)\psi' + (x+1/2)\psi\right]} \tag{A.25}$$

in which

$$\boxed{x \equiv \frac{(\mathbf{v}-\mathbf{V}_{s'})^2}{v_{Ts'}^2}.} \tag{A.26}$$

Next, we evaluate $H_{s'}(\mathbf{v})$:

$$H_{s'}(\mathbf{v}) = \frac{m_s}{m_{ss'}}\int d^3v' \frac{f_{s'}}{u} = \frac{m_s}{m_{ss'}}\frac{n_{s'}}{\pi^{3/2}v_{Ts'}^3}\int d^3u \frac{1}{u}\exp\left[-\frac{(\mathbf{u}+\mathbf{w}_{s'})^2}{v_{Ts'}^2}\right]. \tag{A.27}$$

Again, using spherical coordinates and the substitution $x = \cos\theta$, this is

$$\begin{aligned} H_{s'}(\mathbf{v}) &= \frac{m_s}{m_{ss'}}\frac{2\pi n_{s'}}{\pi^{3/2}v_{Ts'}^3}\int_0^{\infty} du\, u e^{-(u^2+w_{s'}^2)^2/v_{Ts'}^2}\int_{-1}^{1} dx\, e^{-2uw_{s'}x/v_{Ts'}^2} \\ &= -\frac{m_s}{m_{ss'}}\frac{n_{s'}}{\sqrt{\pi}v_{Ts'}w_{s'}}\int_0^{\infty} du\left[e^{-(u+w)^2/v_{Ts'}^2} - e^{-(u+w_{s'})^2/v_{Ts'}^2}\right]. \end{aligned} \tag{A.28}$$

Using the substitution $y = (u + w_{s'})/v_{Ts'}$ into the first integral and $y = -(u - w_{s'})/v_{Ts'}$ into the second gives

$$H_{s'}(\mathbf{v}) = -\frac{m_s}{m_{ss'}} \frac{n_{s'}}{\sqrt{\pi} w_{s'}} \left[\int_{\sqrt{x}}^{\infty} dy\, e^{-y^2} + \int_{\sqrt{x}}^{-\infty} dy\, e^{-y^2} \right]. \tag{A.29}$$

Rearranging the limits of the integrand, this simplifies to

$$H_{s'}(\mathbf{v}) = \frac{m_s}{m_{ss'}} \frac{n_{s'}}{w_{s'}} \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} dy e^{-y^2}, \tag{A.30}$$

which can be written in terms of the error function

$$H_{s'}(\mathbf{v}) = \frac{m_s}{m_{ss'}} \frac{n_{s'}}{v_{Ts'}} \frac{1}{\sqrt{x}} \mathrm{erf}\big(\sqrt{x}\big), \tag{A.31}$$

or the Maxwell integral

$$\boxed{H_{s'}(\mathbf{v}) = \frac{m_s}{m_{ss'}} \frac{n_{s'}}{v_{Ts'}} \frac{\psi + \psi'}{\sqrt{x}}.} \tag{A.32}$$

Appendix B

Kinetic Theory With Equilibrium Fields

In deriving a collision operator in chapter 2, we assumed that equilibrium electric and magnetic fields were negligible. Consequently, the collisionless particle trajectories were simply straight lines at constant speed. In the applications described in chapters 4, 5 and 6, however, we apply the collision operator of chapter 2 to the presheath region of a plasma where there is a weak equilibrium electric field that accelerates particles. Furthermore, weak equilibrium magnetic fields may be present from the ambient field of the earth, as well as from currents generated by ion flow in the presheath.

In this appendix we consider collision operators that include effects of equilibrium electric and magnetic fields. We still assume that the only instabilities present are electrostatic. When considering effects of an equilibrium magnetic field, we also assume that the field is sufficiently uniform that it can be approximated by a constant value in a single Cartesian direction. We apply the method of characteristics in addition to the Fourier-Laplace transforms used in chapter 2. For the constant magnetic field, the characteristic trajectories are helices centered about the magnetic filed direction.

The resultant collision operators show that an equilibrium electric field significantly modifies the collision operator only when the field is strong. In particular, when the gradient scale length of the equilibrium potential variation is comparable to a Debye length, or the wavelength of the unstable waves (whichever is longer). Thus the weak field of a presheath does not significantly modify the collision operator of chapter 2 for the relevant micro-instabilities. In fact, we will find that if the equilibrium electric field is strong enough to significantly modify the collision operator, this implies that the ionized medium is not quasineutral. In this case the ionized gas is no longer a plasma according to the conventional definition. In contrast, the presence of an equilibrium magnetic field can significantly

modify the collision operator in a state where the plasma can remain quasineutral. These modifications may be interesting for many applications where very strong magnetic fields are applied to plasmas, but we show here that the Earth's magnetic field and the magnetic fields generated by driven currents in typical presheaths are sufficiently weak that they do not significantly modify the collision operator of chapter 2. Thus, the collision operator of chapter 2 that neglected equilibrium field effects can be shown to be valid even in the presence of the weak fields found in the applications of chapters 4, 5 and 6.

B.1 For a General Field Configuration

Recall from equations 2.5 and 2.8 of section 2.1.2 that the kinetic equation for a species s including equilibrium electric and magnetic fields is given by

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s}\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = C(f_s) = -\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{J}_v \tag{B.1}$$

in which

$$\mathbf{J}_v \equiv \frac{q_s}{m_s}\left\langle \left(\delta\mathbf{E} + \frac{\mathbf{v}}{c} \times \delta\mathbf{B}\right) \delta f_s \right\rangle \tag{B.2}$$

is the collisional current. The collisional current is determined by the linearized $\mathcal{O}(\delta)$ equation

$$\frac{\partial \delta f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial \delta f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s}\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \frac{\partial \delta f_s}{\partial \mathbf{v}} = -\frac{q_s}{m_s}\left(\delta\mathbf{E} + \frac{\mathbf{v}}{c} \times \delta\mathbf{B}\right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} \tag{B.3}$$

along with Gauss's law

$$\frac{\partial}{\partial \mathbf{x}} \cdot \delta\mathbf{E} = 4\pi \sum_s q_s \int d^3v \, \delta f_s. \tag{B.4}$$

To proceed, we will use the method of characteristics. We first prime all of the $(\mathbf{x}, \mathbf{v}, t)$ coordinates in equations B.3 and B.4 to distinguish them from later "end point" values denoted without the primes. The characteristics are the collisionless trajectories of single particles. For a general equilibrium forcing function $\mathbf{F}$ these are given by

$$\frac{d\mathbf{x}'}{dt'} = \mathbf{v}' \quad \text{and} \quad \frac{d\mathbf{v}'}{dt'} = \frac{\mathbf{F}}{m_s} \tag{B.5}$$

subject to the "end point" conditions

$$\mathbf{x}'(t' = t) = \mathbf{x} \quad \text{and} \quad \mathbf{v}'(t' = t) = \mathbf{v}. \tag{B.6}$$

In general $\mathbf{F}$ is a function of $(\mathbf{v}', \mathbf{E}, \mathbf{B}, \bar{\mathbf{x}}, \bar{\mathbf{t}})$ where $\bar{\mathbf{x}}$ and $\bar{\mathbf{t}}$ are the space and time-scales for variation of the equilibrium fields. Note that equilibrium gravitational forces can also be easily included within this framework.

Integrating the $\mathbf{x}'$ evolution equation leads to an expression of the form $\mathbf{x}' = \mathbf{x} + \mathbf{d}$ where $\mathbf{d}$ depends on the forcing function $\mathbf{F}$ and the same variables $(\mathbf{v}', \mathbf{E}, \mathbf{B}, \bar{\mathbf{x}}, \bar{\mathbf{t}})$. We will consider specific cases for $\mathbf{d}$ in sections B.2 and B.3 where the forcing function is from equilibrium electric and magnetic fields respectively. For now, we will not specify the particular form for $\mathbf{F}$ or $\mathbf{d}$.

Writing equation B.3 in terms of primed variables gives

$$\frac{d\delta f(\mathbf{x}', \mathbf{v}', t')}{dt'} = -\frac{q_s}{m_s}\frac{\partial f_s}{\partial \mathbf{v}} \cdot \delta\mathbf{E}(\mathbf{x}', t'). \tag{B.7}$$

Note that the $\partial f_s/\partial \mathbf{v}$ term is not written in the primed variables because it is a constant on the short space and time scales of δf and $\delta\mathbf{E}$. We apply the so-called "end point" condition for the characteristics that $\mathbf{x}'(t' = t) = \mathbf{x}$ and $\mathbf{v}'(t' = t) = \mathbf{v}$. Integrating from $t' = 0$ to t gives

$$\delta f_s(\mathbf{x}, \mathbf{v}, t) = \delta f_s(\mathbf{x}', \mathbf{v}', t' = 0) - \frac{q_s}{m_s}\frac{\partial f_s}{\partial \mathbf{v}} \cdot \int_0^t dt' \delta\mathbf{E}(\mathbf{x}', t'). \tag{B.8}$$

We next apply the Fourier-Laplace transforms, defined by equations 2.14 and 2.15, to each term in equation B.8. The left side will simply give $\delta\hat{f}_s(\mathbf{k}, \mathbf{v}, \omega)$. The first term on the right is

$$\delta\hat{f}_s(\mathbf{k}, \mathbf{v}', t' = 0) = \int_0^\infty dt \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x} + i\omega t} \delta f_s(\mathbf{x}', \mathbf{v}', t' = 0). \tag{B.9}$$

Inserting our characteristic equation $\mathbf{x}' = \mathbf{x} + \mathbf{d}$ gives

$$\begin{aligned} \delta\hat{f}_s(\mathbf{k}, \mathbf{v}, t' = 0) &= \int_0^\infty \int d^3x \exp\big[-i\mathbf{k}\cdot\mathbf{x}' + i\mathbf{k}\cdot\mathbf{d}(t' = 0) + i\omega t\big] \delta f_s(\mathbf{x}', \mathbf{v}, t' = 0) \qquad \text{(B.10)} \\ &= \underbrace{\int d^3x' e^{-i\mathbf{k}\cdot\mathbf{x}'} \delta f_s(\mathbf{x}', \mathbf{v}, t' = 0)}_{\equiv \delta\tilde{f}_s(\mathbf{k}, \mathbf{v}, t' = 0)} \underbrace{\int_0^\infty dt \exp\big[+i\mathbf{k}\cdot\mathbf{d}(t' = 0) + i\omega t\big]}_{= \int_0^\infty d\tau \exp[i\mathbf{k}\cdot\mathbf{d} + i\omega\tau]} \end{aligned}$$

in which we've used $\tau \equiv t - t'$. We then have

$$\delta\hat{f}_s(\mathbf{k}, \mathbf{v}, t' = 0) = \frac{i\delta\tilde{f}_s(\mathbf{k}, \mathbf{v}, t' = 0)}{\bar{\omega}_p} \tag{B.11}$$

in which

$$\frac{1}{\bar{\omega}_p} \equiv -i \int_0^\infty d\tau \exp\big[i\big(\mathbf{k}\cdot\mathbf{d} + \omega\tau\big)\big]. \tag{B.12}$$

Without equilibrium fields $\mathbf{d} = \mathbf{v}\tau$ and we get $\bar{\omega}_p = \omega - \mathbf{k}\cdot\mathbf{v}$, which is a simple way to connect with the results of chapter 2.

Next consider the electric field fluctuation term of equation B.8;

$$\int_0^t dt' \delta\mathbf{E}(\mathbf{x}', t') = \int_0^t dt' \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{x}' - \omega t')} \delta\hat{\mathbf{E}}(\mathbf{k}, \omega). \tag{B.13}$$

Noting that $\exp[i(\mathbf{k}\cdot\mathbf{x}' - \omega t')] = \exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)] \exp[i\mathbf{k}\cdot\mathbf{d} + i\omega\tau]$ gives

$$\int_0^t dt' \delta\mathbf{E}(\mathbf{x}', t') = \underbrace{\int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \delta\hat{\mathbf{E}}(\mathbf{k}, \omega)}_{\delta\mathbf{E}(\mathbf{x}, t)} \underbrace{\int_0^t dt' \exp\left[i\mathbf{k}\cdot\mathbf{d} + i\omega(t - t')\right]}_{= \int_0^t d\tau \exp[i\mathbf{k}\cdot\mathbf{d} + i\omega\tau]}. \tag{B.14}$$

However, on the short timescale of $\delta\mathbf{E}$, we can take $t \to \infty$ in the last integral, so we find

$$\int_0^t dt' \delta\mathbf{E}(\mathbf{x}', t') \approx \frac{i\delta\mathbf{E}(\mathbf{x}, t)}{\bar{\omega}_p}. \tag{B.15}$$

Putting these terms into the transform of equation B.8 gives

$$\delta\hat{f}_s(\mathbf{k}, \mathbf{v}, \omega) = \frac{i\delta\tilde{f}_s(\mathbf{k}, \mathbf{v}, t' = 0)}{\bar{\omega}_p} - \frac{q_s}{m_s} \frac{i\delta\hat{\mathbf{E}}(\mathbf{k}, \omega)}{\bar{\omega}_p} \cdot \frac{\partial f_s}{\partial \mathbf{v}}. \tag{B.16}$$

Putting equation B.16 into Gauss's law

$$k^2 \delta\hat{\phi} = 4\pi \sum_s q_s \int d^3v \delta\hat{f}_s(\mathbf{k}, \mathbf{v}, \omega), \tag{B.17}$$

where we have applied the electrostatic fluctuation approximation $\delta\hat{\mathbf{E}}(\mathbf{k}, \omega) = -i\mathbf{k}\delta\hat{\phi}(\mathbf{k}, \omega)$, gives

$$\delta\hat{\phi}(\mathbf{k}, \omega) = \sum \frac{4\pi q_s}{k^2 \hat{\varepsilon}(\mathbf{k}, \omega)} \int d^3v \frac{i\delta\tilde{f}_s(\mathbf{k}, \mathbf{v}, t' = 0)}{\bar{\omega}_p} \tag{B.18}$$

in which

$$\hat{\varepsilon}(\mathbf{k}, \omega) = 1 + \sum_s \frac{4\pi q_s^2}{k^2 m_s} \int d^3v \frac{\mathbf{k}\cdot\partial f_s/\partial \mathbf{v}}{\bar{\omega}_p} \tag{B.19}$$

is the plasma dielectric function.

Next, we insert the discrete particle initial condition

$$\delta\tilde{f}_s(\mathbf{k}, \mathbf{v}, t' = 0) = \sum_{i=1}^{N} e^{-i\mathbf{k}\cdot\mathbf{x}_{io}} \delta(\mathbf{v} - \mathbf{v}_{io}) - (2\pi)^3 \delta(\mathbf{k}) f_s \tag{B.20}$$

into equations B.16 and B.18. First, in equation B.18 this gives

$$\delta\hat{\phi}(\mathbf{k}, \omega) = \sum_s \frac{4\pi q_s}{k^2 \hat{\varepsilon}(\mathbf{k}, \omega)} \int d^3v \sum_{i=1}^{N} \frac{ie^{-i\mathbf{k}\cdot\mathbf{x}_{io}} \delta(\mathbf{v} - \mathbf{v}_{io})}{\bar{\omega}_p} - \sum_s \frac{4\pi q_s}{k^2 \hat{\varepsilon}} \int d^3v \frac{i(2\pi)^3 \delta(\mathbf{k}) f_s}{\bar{\omega}_p}. \tag{B.21}$$

However, the last term vanishes due to quasineutrality. To see this, first notice that

$$\frac{\delta(\mathbf{k})}{\bar{\omega}_p} = \delta(\mathbf{k})(-i)\int_0^\infty d\tau e^{-i\mathbf{k}\cdot\mathbf{d}-i\omega\tau} \to \delta(\mathbf{k})(-i)\int_0^\infty d\tau e^{-i\omega\tau} = \frac{2\pi\delta(\mathbf{k})}{\omega}. \tag{B.22}$$

With this, we find

$$\sum_s \frac{4\pi q_s}{k^2\hat{\varepsilon}}\int d^3v \frac{i(2\pi)^3\delta(\mathbf{k})f_s}{\bar{\omega}_p} \to \frac{(4\pi)(2\pi)^4}{k^2\hat{\varepsilon}}\frac{\delta(\mathbf{k})}{\omega}\underbrace{\sum_s q_s n_s}_{=0} = 0 \tag{B.23}$$

and we are left with

$$\delta\hat{\phi}(\mathbf{k},\omega) = \sum_{s,i=1}^{N}\frac{4\pi q_s}{k^2\hat{\varepsilon}(\mathbf{k},\omega)}\frac{ie^{-i\mathbf{k}\cdot\mathbf{x}_{io}}}{\bar{\omega}_p(\mathbf{v}=\mathbf{v}_{io})}. \tag{B.24}$$

Putting the discrete particle term in equation B.16 gives

$$\delta\hat{f}_s(\mathbf{k},\omega) = \sum_{s',i=1}\left[\frac{ie^{-i\mathbf{k}\cdot\mathbf{x}_{io}}\delta(\mathbf{v}-\mathbf{v}_{io})}{\bar{\omega}_p} - \frac{i(2\pi)^3\delta(\mathbf{k})f_s}{\bar{\omega}_p} - \frac{4\pi q_s q_{s'}}{m_s k^2\hat{\varepsilon}(\mathbf{k},\omega)}\frac{i\mathbf{k}\cdot\partial f_s/\partial\mathbf{v}}{\bar{\omega}_p}\frac{e^{-i\mathbf{k}\cdot\mathbf{x}_{io}}}{\bar{\omega}_p(\mathbf{v}=\mathbf{v}_{io})}\right]. \tag{B.25}$$

Using equation B.25 along with

$$\delta\hat{\mathbf{E}}(\mathbf{k}',\omega') = \sum_{s'',l=1}\frac{4\pi q_{s''}}{k'^2\hat{\varepsilon}(\mathbf{k}',\omega')}\frac{\mathbf{k}'e^{-i\mathbf{k}'\cdot\mathbf{x}_{lo}}}{\bar{\omega}'_p(\mathbf{v}=\mathbf{v}_{lo})} \tag{B.26}$$

we will calculate the collisional current. Note that here $\bar{\omega}'_p = \bar{\omega}_p(\mathbf{k}',\omega')$.

The transform of the collisional current is

$$\hat{\mathbf{J}}^s_v(\mathbf{k},\mathbf{k}',\mathbf{v},\omega,\omega') = \frac{q_s}{m_s}\left\langle \delta\hat{\mathbf{E}}(\mathbf{k}',\omega')\delta\hat{f}_s(\mathbf{k},\mathbf{v},\omega)\right\rangle \tag{B.27}$$

and putting in the above gives

$$\begin{aligned}
\hat{\mathbf{J}}^s_v &= \frac{q_s}{m_s}\left\langle \sum_{s'',l=1}\frac{4\pi q_{s''}}{k'^2\hat{\varepsilon}(\mathbf{k}',\omega')}\frac{\mathbf{k}'e^{-i\mathbf{k}'\cdot\mathbf{x}_{lo}}}{\bar{\omega}'_p(\mathbf{v}=\mathbf{v}_{lo})}\sum_{i=1}^{N}\frac{ie^{-i\mathbf{k}\cdot\mathbf{x}_{io}}\delta(\mathbf{v}-\mathbf{v}_{io})}{\bar{\omega}_p}\right\rangle \to \text{term 1} \qquad \text{(B.28)}\\
&- \frac{q_s}{m_s}\left\langle \sum_{s'',l=1}\frac{4\pi q_{s''}}{k'^2\hat{\varepsilon}(\mathbf{k}',\omega')}\frac{\mathbf{k}'e^{-i\mathbf{k}'\cdot\mathbf{x}_{lo}}}{\bar{\omega}'_p(\mathbf{v}=\mathbf{v}_{lo})}\frac{i(2\pi)^3\delta(\mathbf{k})f_s}{\bar{\omega}_p}\right\rangle \to \text{term 2}\\
&- \frac{q_s}{m_s}\left\langle \sum_{s'',l=1}\frac{4\pi q_{s''}}{k'^2\hat{\varepsilon}(\mathbf{k}',\omega')}\frac{\mathbf{k}'e^{-i\mathbf{k}'\cdot\mathbf{x}_{lo}}}{\bar{\omega}'_p(\mathbf{v}=\mathbf{v}_{lo})}\sum_{s',i=1}^{N}\frac{4\pi q_s q_{s'}}{m_s k^2\hat{\varepsilon}(\mathbf{k},\omega)}\frac{i\mathbf{k}\cdot\partial f_s/\partial\mathbf{v}}{\bar{\omega}_p}\frac{e^{-i\mathbf{k}\cdot\mathbf{x}_{io}}}{\bar{\omega}_p(\mathbf{v}=\mathbf{v}_{io})}\right\rangle \to \text{term 3.}
\end{aligned}$$

We will consider each of these three terms individually. Recall from equation 2.25 that the definition of ensemble average is

$$\langle\ldots\rangle \equiv \prod_{j=1}^{N}\int d^3x_{jo}d^3v_{jo}\frac{f(\mathbf{v}_{jo})}{nV}(\ldots). \tag{B.29}$$

Considering term 1. For $i \neq l$, the $\int d^3x_{lo}$ integral will give a $\delta(\mathbf{k}')$; but these terms $\to 0$ in the limit that $\mathbf{k}' \to 0$, so the unlike particle terms vanish. We are then left with the like particle terms for which $s'' = s$, and we have

$$1 = \frac{q_s}{m_s} \underbrace{\frac{N}{nV}}_{=1} \int d^3x_o d^3v_o \frac{4\pi i q_s}{k'^2 \hat{\varepsilon}(\mathbf{k}', \omega')} \frac{\mathbf{k}' e^{-i\mathbf{k}'\cdot\mathbf{x}_o}}{\bar{\omega}_p'(\mathbf{v} = \mathbf{v}_o)} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}_o}}{\bar{\omega}_p} \delta(\mathbf{v} - \mathbf{v}_o) f_s(\mathbf{v}_o). \tag{B.30}$$

Using

$$\int d^3x_o e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}_o} = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \tag{B.31}$$

and changing the dummy variable $\mathbf{v}_o$ to $\mathbf{v}'$, term 1 can be written

$$1 = \frac{4\pi q_s^2}{m_s k^2} \int d^3v' \frac{i\mathbf{k}'(2\pi)^3 \delta(\mathbf{k} + \mathbf{k}')}{\bar{\omega}_p'(\mathbf{v} = \mathbf{v}')\, \bar{\omega}_p \hat{\varepsilon}(\mathbf{k}', \omega')} \delta(\mathbf{v} - \mathbf{v}') f_s(\mathbf{v}'). \tag{B.32}$$

Term 2 vanishes for the same reason as the unlike particle terms in 1 did. That is, because the x_l integral gives a $\delta(\mathbf{k}')$ and the term $\to 0$ in the limit $\mathbf{k}' \to 0$.

The unlike particle terms $i \neq l$ of term 3 vanish for the same reason as they do in term 1. This also implies that only $s' = s''$ terms survive. Performing the x_o integral, we find

$$3 = -\frac{4\pi q_s^2}{m_s k^2} \int d^3v' \frac{i\mathbf{k}'(2\pi)^3 \delta(\mathbf{k} + \mathbf{k}')}{\hat{\varepsilon}(\mathbf{k}', \omega')\, \bar{\omega}_p'(\mathbf{v} = \mathbf{v}') \bar{\omega}_p} f_s(\mathbf{v}') \sum_{s'} \frac{4\pi q_{s'}^2}{k^2 \hat{\varepsilon}(\mathbf{k}, \omega)} \frac{\mathbf{k} \cdot \partial f_s / \partial \mathbf{v}}{m_s \bar{\omega}_p(\mathbf{v} = \mathbf{v}')}. \tag{B.33}$$

We are then left with the following expression for the collisional current

$$\hat{\mathbf{J}}_v^s = \frac{4\pi q_s^2}{m_s k^2} \int d^3v' \frac{i\mathbf{k}'(2\pi)^3 \delta(\mathbf{k} + \mathbf{k}')}{\hat{\varepsilon}(\mathbf{k}', \omega')\ \bar{\omega}_p'(\mathbf{v} = \mathbf{v}')\ \bar{\omega}_p} f_s(\mathbf{v}') \left[\delta(\mathbf{v} - \mathbf{v}') - \sum_{s'} \frac{4\pi q_{s'}^2}{m_s k^2} \frac{\mathbf{k} \cdot \partial f_s / \partial \mathbf{v}}{\hat{\varepsilon}(\mathbf{k}, \omega) \bar{\omega}_p(\mathbf{v} = \mathbf{v}')} \right]. \tag{B.34}$$

Doing the trivial $\mathbf{v}'$ integral in the first term and multiplying this term by $\hat{\varepsilon}/\hat{\varepsilon}$ gives $\hat{\mathbf{J}}_v^s = \sum_{s'} \hat{\mathbf{J}}_v^{s/s'}$ in which

$$\hat{\mathbf{J}}_v^{s/s'} = \frac{(4\pi)^2 q_s^2 q_{s'}^2}{m_s k^4} \int d^3v' \frac{i\mathbf{k}'(2\pi)^3 \delta(\mathbf{k} + \mathbf{k}')}{\hat{\varepsilon}(\mathbf{k}', \omega') \hat{\varepsilon}(\mathbf{k}, \omega) \bar{\omega}_p\, \bar{\omega}_p(\mathbf{v} = \mathbf{v}')} \left[\frac{f_s(\mathbf{v}) \mathbf{k} \cdot \partial f_{s'} / \partial \mathbf{v}'}{m_{s'} \bar{\omega}_p'(\mathbf{v} = \mathbf{v})} - \frac{f_{s'}(\mathbf{v}') \mathbf{k} \cdot \partial f_s / \partial \mathbf{v}}{m_s \bar{\omega}_p'(\mathbf{v} = \mathbf{v}')} \right] \tag{B.35}$$

$$+ \frac{4\pi q_s^2}{k^2 m_s} \frac{i\mathbf{k}'(2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') f_s(\mathbf{v})}{\hat{\varepsilon}(\mathbf{k}', \omega') \bar{\omega}_p \bar{\omega}_p'(\mathbf{v} = \mathbf{v}) \hat{\varepsilon}(\mathbf{k}, \omega)}.$$

The last term will vanish upon inverse Fourier transforming due to odd parity in $\mathbf{k}$.

We can then write

$$\boxed{\begin{aligned} \hat{\mathbf{J}}_v^{s/s'} = \frac{(4\pi)^2 q_s^2 q_{s'}^2}{m_s k^4} \int d^3v' \frac{i\mathbf{k}'(2\pi)^3 \delta(\mathbf{k} + \mathbf{k}')}{\hat{\varepsilon}(\mathbf{k}', \omega') \hat{\varepsilon}(\mathbf{k}, \omega) \bar{\omega}_p\, \bar{\omega}_p(\mathbf{v} = \mathbf{v}') \bar{\omega}_p'(\mathbf{v} = \mathbf{v}) \bar{\omega}_p'(\mathbf{v} = \mathbf{v}')} \\ \times \left[\bar{\omega}_p'(\mathbf{v} = \mathbf{v}') \frac{f_s(\mathbf{v}) \mathbf{k} \cdot \partial f_{s'} / \partial \mathbf{v}'}{m_{s'}} - \bar{\omega}_p'(\mathbf{v} = \mathbf{v}) \frac{f_{s'}(\mathbf{v}') \mathbf{k} \cdot \partial f_s / \partial \mathbf{v}}{m_s} \right] \end{aligned}} \tag{B.36}$$

which is almost in the Landau form. Usually we have the $\bar{\omega}_p$ terms in the square brackets $\rightarrow \omega'$ because the terms $\propto \mathbf{k}$ have odd parity and vanish. With equilibrium magnetic fields present this is not so obvious (and may lead to complications).

It is difficult to extract much more information from the expression for collisional current given by equation B.36 without specifying a particular forcing function. This is because $1/\bar{\omega}_p$ needs to be specified before the Fourier-Laplace transforms can be inverted to give an expression for a collision operator in real space and time. In the next two sections we will consider particular forcing functions and thus specify $1/\bar{\omega}_p$, which leads to more explicit formulations of the collision operator and plasma dielectric function.

B.2 With an Equilibrium Electric Field

So far, we have assumed that "equilibrium" field quantities (e.g., $\mathbf{E}$) vary over much longer space and timescales than perturbed quantities (e.g., $\delta\mathbf{E}$). We denote the long spatial scale l. The perturbed quantities vary in space on the characteristic scale $\delta l \sim 1/k$. For a stable plasma, k typically ranges from $1/b_{\min}$ to $1/\lambda_{De}$. For an unstable plasma, k ranges over all unstable wavenumbers, which for the instabilities considered in this work are on the order of $1/\lambda_{De}$ and have the approximate range $k \sim 1/\lambda_{De} - 100/\lambda_{De}$. The uniformity condition of the equilibrium electrostatic potential can be expressed as

$$\frac{\delta l}{l} \sim \frac{1/k}{1/(d\ln\phi/d\bar{x})} = \frac{1}{k\phi}\frac{d\phi}{d\bar{x}} \ll 1. \tag{B.37}$$

In this section, our fundamental scaling assumption of equilibrium electric fields given by equation B.37 will be important for evaluating the ultimate contribution of the equilibrium fields to modifying the collision operator and linear wave properties derived in chapter 2. It is also noteworthy that equation B.37 must be satisfied in order for quasineutrality to hold. We have already assumed in equation B.23 that the plasma is quasineutral.

For an equilibrium electric field, the forcing function is $\mathbf{F} = q_s\mathbf{E}$, so the characteristic equation B.5 is $d\mathbf{v}'/dt' = q_s\mathbf{E}/m_s$. Integrating this over t' and enforcing the "end point" condition $\mathbf{v}'(t'=t) = \mathbf{v}$ gives $\mathbf{v}' = \mathbf{v} - \frac{q_s}{m_s}\mathbf{E}\tau$ where $\tau \equiv t - t'$. Integrating this and enforcing $\mathbf{x}'(t'=t) = \mathbf{x}$ gives $\mathbf{x}' = \mathbf{x} - \mathbf{v}\tau + \frac{1}{2}\frac{q_s}{m_s}\mathbf{E}\tau^2$.

Thus we can write $\mathbf{x}'$ in the general form $\mathbf{x}' = \mathbf{x} + \mathbf{d}$ where

$$\mathbf{d} = -\mathbf{v}\tau + \frac{1}{2}\frac{q_s}{m_s}\mathbf{E}\tau^2. \tag{B.38}$$

Putting the characteristic of equation B.38 into equation B.12 gives

$$\frac{1}{\bar{\omega}_p} = -i\int_0^\infty d\tau\, \exp\{i[(\omega - \mathbf{k}\cdot\mathbf{v})\tau + \omega_E^2\tau^2]\} \tag{B.39}$$

in which

$$\omega_E^2 \equiv \frac{1}{2}\frac{q_s}{m_s}\mathbf{k}\cdot\mathbf{E} \tag{B.40}$$

is defined for notational convenience. The magnitude of the electric field effects can be estimated by applying the substitution $w = (\omega - \mathbf{k}\cdot\mathbf{v})\tau$, which gives

$$\frac{1}{\bar{\omega}_p} = \frac{-i}{\omega - \mathbf{k}\cdot\mathbf{v}}\int_0^\infty dw\, \exp\left\{i\left[w + \frac{\omega_E^2}{(\omega - \mathbf{k}\cdot\mathbf{v})^2}w^2\right]\right\}. \tag{B.41}$$

Thus, we find that the importance of electric field effects is associated with the size of $\omega_E^2/(\omega - \mathbf{k}\cdot\mathbf{v})^2$. However,

$$\frac{\omega_E^2}{(\omega - \mathbf{k}\cdot\mathbf{v})^2} \sim \frac{1}{2}\frac{q_s}{m_s}\frac{kE}{k^2 v_{Ts}^2} \sim \frac{1}{2}\frac{q_s\phi}{T_s}\frac{1}{k\phi}\frac{d\phi}{dx} \ll 1 \tag{B.42}$$

which must be small due to the uniformity condition on E from equation B.37. For the weak uniform fields of a presheath we find

$$\frac{1}{k\phi}\frac{d\phi}{dx} \sim \frac{\lambda_{De}}{l} \sim 10^{-4}, \tag{B.43}$$

which is extremely small. Here l is the presheath length scale which is typically thousands or tens of thousands of Debye lengths. Thus, equation B.39 gives $\bar{\omega}_p \approx \omega - \mathbf{k}\cdot\mathbf{v}$ and the electric field free results are returned. Note also that if the equilibrium electric field is strong enough to modify the collision operator, it implies that the quasineutrality condition of equation B.37 is violated.

Although equation B.43 shows that the weak presheath electric fields do not significantly affect the collision operator, it may still be useful to obtain the order of corrections due to the field. Equation B.41 can be written explicitly in terms of exponential and complimentary error functions

$$\frac{1}{\bar{\omega}_p} = \frac{1}{\omega - \mathbf{k}\cdot\mathbf{v}}\sqrt{\pi}w_E \exp(w_E^2)\,\mathrm{erfc}(w_E) \tag{B.44}$$

in which we have defined

$$w_E \equiv \frac{\sqrt{-i}(\omega - \mathbf{k}\cdot\mathbf{v})}{2\omega_E}. \tag{B.45}$$

Putting equation B.44 into equation B.36 provides an explicit expression for the collisional current, and hence collision operator, in the presence of a weak equilibrium electric field. For the quasineutral plasmas of interest, $w_E \gg 1$. Applying the large argument ($|w_E| \gg 1$) expansion for the complimentary error function

$$\operatorname{erfc}(w_E) = \frac{\exp(-w_E^2)}{\sqrt{\pi} w_E}\left(1 - \frac{1}{2w_E^2} - \dots\right) \tag{B.46}$$

to equation B.44, we find

$$\frac{1}{\bar{\omega}_p} = \frac{1}{\omega - \mathbf{k}\cdot\mathbf{v}}\left(1 - \frac{2i\omega_E^2}{(\omega - \mathbf{k}\cdot\mathbf{v})^2} + \dots\right). \tag{B.47}$$

Thus, corrections due to the equilibrium electric field are $\mathcal{O}\{\omega_E^2/(\omega - \mathbf{k}\cdot\mathbf{v})^2\}$, which is small for quasineutral plasmas.

B.3 With a Uniform Equilibrium Magnetic Field

For a straight-line magnetic field, $\mathbf{B}$, the characteristic particle trajectories are [33]

$$\mathbf{d} \equiv \frac{v_\perp}{\Omega_s}\{\sin\varphi - \sin[\varphi + \Omega_s\tau]\}\hat{x} - \frac{v_\perp}{\Omega_s}\{\cos\varphi - \cos[\varphi + \Omega_s\tau]\}\hat{y} - v_z\tau\hat{z}. \tag{B.48}$$

Recall that

$$\frac{1}{\bar{\omega}_p} \equiv -i\int_0^\infty d\tau \exp[-i\mathbf{k}\cdot\mathbf{d} - i\omega\tau]. \tag{B.49}$$

In this case, we find that in the parameter of interest is the size of the gyro-radius compared to k. Recall that the gyro-radius is defined as

$$\rho_s \equiv \frac{v_\perp}{\Omega_s} = \frac{c\, m_s v_\perp}{q_s B}. \tag{B.50}$$

Since $1/\tau \sim k v_\perp$, we find

$$d \sim \rho_s\{\sin\varphi - \sin[\varphi + 1/(k\rho_s)]\}\hat{x} - \frac{v_\perp}{\Omega_s}\{\cos\varphi - \cos[\varphi + 1/(k\rho_s)]\}\hat{y} - v_z\tau\hat{z}. \tag{B.51}$$

For the low magnetic fields of interest in this work, and for the wavelengths characteristic of a Debye length, we find

$$k\rho_s \sim \frac{\rho_s}{\lambda_{De}} \gg 1. \tag{B.52}$$

Thus, expanding d for $1/(k\rho_s) \ll 1$, yields $\mathbf{d} \approx \mathbf{v}\tau$. With this, we find $\bar{\omega}_p \approx (\omega - \mathbf{k}\cdot\mathbf{v})$ and the magnetic field free results are returned.

Hence, we find that the characteristic kinetic scale of interest (which is the Debye length for conventional Coulomb interactions, or the wavelength of the unstable mode for instability-enhanced interactions) must be of comparable magnitude to the gyroradius (for whatever particles one is interested in calculating a collision operator for) before magnetic fields significantly affect the collision operator. This could be accomplished by extremely strong fields or long wavelength instabilities. However, neither of these are found in the plasmas of interest in this work, so magnetic field corrections are negligible here.

Although the magnetic field corrections are negligible for the essentially unmagnetized plasmas of interest in this work, equation B.49 can be evaluated explicitly in terms of Bessel functions for a uniform magnetic field. To show this, we first note the Jacobi-Anger expansion

$$e^{\pm iz\cos\theta} = \sum_{n=-\infty}^{\infty} (\pm i)^n J_n(z) e^{in\theta} \quad \text{and} \quad e^{\pm iz\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(z) e^{\pm in\theta}, \tag{B.53}$$

which can be proven by expanding $e^{iz\sin\theta}$ in a Fourier series, then identifying the Fourier coefficients as the Bessel functions using the integral representation

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\tau - x\sin\tau)} d\tau. \tag{B.54}$$

The second form can be obtained from the first by writing $\sin\theta = \cos(\theta - \pi/2)$ and using the symmetry relation

$$J_n(z) = (-1)^n J_n(-z). \tag{B.55}$$

Putting equation B.53 into equation B.49, yields

$$\frac{1}{\bar{\omega}_p} = -i\exp[i(k_x\rho_s\sin\varphi - k_y\rho_s\cos\varphi)]\int_0^{\infty} d\tau \exp\{[-k_x\rho_s\sin(\varphi+\Omega_s\tau) + k_y\rho_s\cos(\varphi+\omega_s\tau) - k_z v_z\tau + \omega\tau]\}. \tag{B.56}$$

From the Jacobi-Anger expansions above, we find

$$e^{-ik_x\rho_s\sin(\varphi+\Omega_s\tau)} = \sum_{n=-\infty}^{\infty} J_n(k_x\rho_s) e^{-in\varphi} e^{-in\Omega_s\tau} \tag{B.57}$$

and

$$e^{ik_y\rho_s\cos(\varphi+\Omega_s\tau)} = \sum_{l=-\infty}^{\infty} i^l J_l(k_y\rho_s) e^{il\varphi} e^{il\Omega_s\tau}. \tag{B.58}$$

Applying these yields

$$\frac{1}{\bar{\omega}_p} = -i\sum_{n=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} i^l J_n(k_x\rho_s) J_l(k_y\rho_s) e^{i(l-n)\varphi} e^{i(k_x\rho_s\sin\varphi - k_y\rho_s\cos\varphi)} \int_0^{\infty} d\tau e^{i[\omega - k_z v_z + (l-n)\Omega_s]\tau}. \tag{B.59}$$

After evaluating the τ integral

$$\int_0^\infty d\tau e^{i[\omega - k_z v_z + (l-n)\Omega_s]\tau} = \frac{i}{\omega - k_z v_z + (l-n)\Omega_s}, \tag{B.60}$$

we find an explicit expression for $\bar{\omega}_p$ that includes the effects of a uniform equilibrium magnetic field

$$\frac{1}{\bar{\omega}_p} = \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{i^l J_n(k_x \rho_s) J_l(k_y \rho_s) e^{i(l-n)\varphi} e^{i(k_x \rho_s \sin\varphi - k_y \rho_s \cos\varphi)}}{\omega - k_z v_z + (l-n)\Omega_s}. \tag{B.61}$$

Putting equation B.61 into equation B.36 provides an explicit expression for the collisional current, and hence the collision operator, when a uniform magnetic field is present. For the plasmas of interest in this work, the gyroradius is much larger than the Debye length (which is also approximately the wavelength of the unstable modes of interest), and the magnetic field provides negligible modifications to the unmagnetized plasma collision operator derived in chapter 2.

Appendix C

The Incomplete Plasma Dispersion Function

The incomplete plasma dispersion function is defined as [83]

$$Z(\nu, w) = \frac{1}{\sqrt{\pi}} \int_{\nu}^{\infty} dt \frac{e^{-t^2}}{t - w}. \tag{C.1}$$

It is a useful function to use when calculating the contribution to the plasma dielectric function from a distribution function that can be split in regions of velocity space that are Maxwellians, but may have different temperatures.

C.1 Power Series and Asymptotic Representations

For the power series, $|w| \ll 1$, we can follow the same procedure as with the plasma dispersion function. After applying the Plemelj formula to equation C.1, we find

$$\begin{aligned} Z(\nu, w) &= \frac{1}{\sqrt{\pi}} \mathcal{P} \int_{\nu}^{\infty} \frac{e^{-t^2}}{t - w} dt + i\sqrt{\pi} \int_{\nu}^{\infty} e^{-t^2} \delta(t - w) dt \qquad \text{(C.2)} \\ &= i\sqrt{\pi} e^{-w^2} H(w - \nu) + \frac{1}{\sqrt{\pi}} \mathcal{P} \int_{\nu}^{\infty} \frac{e^{-t^2}}{t - w} dt \end{aligned}$$

where H is the Heaviside step function. After integrating by parts, we find the power series representation of the incomplete plasma dispersion function

$$\begin{aligned} Z(\nu, w) &= i\sqrt{\pi} H(w - \nu) e^{-w^2} + \frac{E_1(\nu^2)}{2\sqrt{\pi}} - \operatorname{erfc}(\nu) w + \frac{1}{\sqrt{\pi}} \left[\frac{e^{-\nu^2}}{2\nu^2} - \frac{E_1(\nu^2)}{2} \right] w^2 \qquad \text{(C.3)} \\ &+ \frac{2}{3} \left\{ \operatorname{erfc}(\nu) - \frac{e^{-\nu^2}}{2\sqrt{\pi}} \left[\frac{1}{\nu} - \frac{1}{\nu^2} \right] \right\} w^3 + \ldots \qquad \text{(C.4)} \end{aligned}$$

Analogously for the asymptotic expansion $|w| \gg 1$, we find

$$\begin{aligned} Z(\nu,w) &\sim i\sigma\sqrt{\pi}H(w_R-\nu)e^{-w^2} - \frac{1}{w}\left\{\frac{\mathrm{erfc}(\nu)}{2} + \frac{1}{2\sqrt{\pi}w}e^{-\nu^2} + \frac{1}{2w^2}\left[\frac{\mathrm{erfc}(\nu)}{2} + \frac{\nu}{\sqrt{\pi}}e^{-\nu^2}\right]\right. \\ &+ \left.\frac{1}{w^3}\frac{1+\nu^2}{2\sqrt{\pi}}e^{-\nu^2} + \ldots\right\} \end{aligned} \tag{C.5}$$

in which

$$\sigma \equiv \begin{cases} 0\ , & \Im\{w\} > 0 \\ 1\ , & \Im\{w\} = 0 \\ 2\ , & \Im\{w\} < 0 \end{cases} . \tag{C.6}$$

The expansions for the conventional plasma dispersion function can be returned by taking the $\nu = -\infty$ limit of these. Doing so yields the power series expansion for $|w| \ll 1$

$$Z(w) = Z(-\infty, w) = i\sqrt{\pi}e^{-w^2} - 2w\left(1 - \frac{2}{3}w^2 + \frac{4}{15}w^4 + \ldots\right) \tag{C.7}$$

and the asymptotic expansion for $|w| \gg 1$

$$Z(w) = Z(-\infty, w) \sim i\sigma\sqrt{\pi}e^{-w^2} - \frac{1}{w}\left(1 + \frac{1}{2w^2} + \frac{3}{4w^4} + \ldots\right). \tag{C.8}$$

C.2 Special Case: $\nu = 0$

The special case $\nu = 0$ can be calculated exactly. This is relevant to plasmas where the distribution function is truncated at $v = 0$, which can occur, for instance, for the electron distribution function near a plasma boundary that is biased more positive than the plasma (i.e., an electron sheath). If $\nu = 0$, equation C.1 is

$$\begin{aligned} Z(0,w) &= e^{-w^2}\left[Z(-iw,0) - Z(0,0)\right] + \frac{1}{2}e^{-w^2}\left[i\sqrt{\pi}\overbrace{\mathrm{erfc}(0)}^{=1}\ \mathrm{erf}(iw) + \frac{E_1(0)}{\sqrt{\pi}}\right] \\ &= e^{-w^2}\left[\frac{1}{\sqrt{\pi}}\int_{-iw}^{\infty} dx\frac{e^{-x^2}}{x} - \frac{1}{\sqrt{\pi}}\int_0^{\infty} dx\frac{e^{-x^2}}{x} + i\frac{\sqrt{\pi}}{2}\mathrm{erf}(iw) + \frac{1}{2\sqrt{\pi}}\int_0^{\infty}\frac{e^{-y}}{y}dy\right]. \end{aligned} \tag{C.9}$$

Notice that the variable change $x^2 = y$ allows the last term to be written

$$\frac{1}{2\sqrt{\pi}}\int_0^{\infty}\frac{e^{-y}}{y}dy = \frac{1}{\sqrt{\pi}}\int_0^{\infty}\frac{e^{-x^2}}{x}dx, \tag{C.10}$$

showing that the second and fourth terms cancel. Noticing also that the first term can be written in terms of an exponential integral

$$\int_{-iw}^{\infty} \frac{e^{-x^2}}{x} dx = -\frac{1}{2} E_1(x^2)\Big|_{-iw}^{\infty}, \tag{C.11}$$

we arrive at the expression

$$Z(0,w) = \frac{e^{-w^2}}{2\sqrt{\pi}} E_1(-w^2) + i\frac{\sqrt{\pi}}{2} e^{-w^2} \mathrm{erf}(iw). \tag{C.12}$$

C.3 Ion-Acoustic Instabilities for a Truncated Maxwellian

In this section, we calculate the ion-acoustic dispersion relation for a plasma with a flowing Maxwellian ion species and an electron species that is Maxwellian except that it is truncated for velocities (in one direction) beyond a critical value, $\mathbf{v}_c$. This situation is pertinent in the Langmuir's paradox application of chapter 4. To find the dispersion relation, we start from the general dielectric function for electrostatic waves in an unmagnetized plasma from equation 2.18

$$\hat{\varepsilon}(\mathbf{k},\omega) = 1 + \sum_s \frac{4\pi q_s^2}{k^2 m_s} \int d^3v \frac{\mathbf{k}\cdot\partial f_s/\partial \mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}}. \tag{C.13}$$

Since we are concerned with electrons in the presheath, we assume no flow-shift in the truncated Maxwellian distribution for electrons. Choosing the Cartesian coordinates (χ,η,ζ) aligned along $\mathbf{k}$, such that $\mathbf{k} = k_\zeta\hat{\zeta} = k\hat{\zeta}$, the truncated Maxwellian distribution for electrons can be written

$$f = H(v_{c,\chi} - v_\chi)H(v_{c,\eta} - v_\eta)H(v_{c,\zeta} - v_\zeta)\frac{n}{\pi^{3/2}v_T^3}\exp\left(-\frac{v_\chi^2 + v_\eta^2 + v_\zeta^2}{v_T^2}\right). \tag{C.14}$$

Since $\mathbf{k}\cdot\partial/\partial\mathbf{v} = k\partial/\partial v_\zeta$, and $\mathbf{k}\cdot\mathbf{v} = kv_\zeta$, the integrals are easier in the χ and η directions. Note that

$$\int_{-\infty}^{v_{c,\chi}} dv_\chi \exp\left(-\frac{v_\chi^2}{v_T^2}\right) = \frac{\sqrt{\pi}v_T}{2}\left[\mathrm{erf}\left(\frac{v_{c,\chi}}{v_T}\right) + 1\right] = \frac{\sqrt{\pi}v_T}{2}\mathrm{erfc}\left(-\frac{v_{c,\chi}}{v_T}\right) \tag{C.15}$$

where the last step comes from $\mathrm{erfc}(z) = 1 - \mathrm{erf}(z)$ and the fact that erf is an odd function $\mathrm{erf}(-z) = -\mathrm{erf}(z)$. Plugging in the analogous formula for the v_η integral we find

$$\frac{4\pi q^2}{k^2 m}\int d^3v \frac{\mathbf{k}\cdot\partial f_s/\partial\mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}} = \frac{\omega_{ps}^2}{k^2\sqrt{\pi}v_T}\frac{1}{4}\mathrm{erfc}\left(-\frac{v_{c,\chi}}{v_T}\right)\mathrm{erfc}\left(-\frac{v_{c,\eta}}{v_T}\right)\int dv_\zeta \frac{\frac{d}{dv_\zeta}H(v_{c,\zeta} - v_\zeta)\exp\left(-\frac{v_\zeta^2}{v_T^2}\right)}{\omega/k - v_\zeta}. \tag{C.16}$$

Next, we evaluate the last integral. To do so, first note that

$$\int dv_\zeta \frac{\frac{d}{dv_\zeta}H(v_{c,\zeta}-v_\zeta)\exp\left(-\frac{v_\zeta^2}{v_T^2}\right)}{\omega/k-v_\zeta} = \int dv_\zeta \frac{-\delta(v_{c,\zeta}-v_\zeta)\exp\left(-\frac{v_\zeta^2}{v_T^2}\right)}{\omega/k-v_\zeta} + \int dv_\zeta \frac{H(v_{c,\zeta}-v_\zeta)\frac{d}{dv_\zeta}\exp\left(-\frac{v_\zeta^2}{v_T^2}\right)}{\omega/k-v_\zeta} \tag{C.17}$$

$$= -\frac{\exp\left(-\frac{v_{c\zeta}^2}{v_T^2}\right)}{\omega/k-v_{c\zeta}} + \int_{-\infty}^{v_{c\zeta}} \frac{\frac{d}{dv_\zeta}\exp\left(-\frac{v_\zeta^2}{v_T^2}\right)}{\omega/k-v_\zeta},$$

which looks like the derivative of a plasma dispersion function, but has a cutoff at the upper limit of integration. Let $t \equiv v_\zeta/v_T$ so $dt = dv_\zeta/v_T$ and let $\nu \equiv v_{c,\zeta}/v_T$ and $w \equiv \omega/kv_T$, then the integral becomes

$$\begin{aligned}\frac{1}{v_T}\int_{-\infty}^{\nu} dt \frac{\frac{d}{dt}e^{-t^2}}{w-t} &= \frac{1}{v_T}\int_{-\infty}^{\nu} dt \frac{-2te^{-t^2}}{w-t} = \frac{1}{v_T}\int_{-\infty}^{\nu} dt\left[\frac{d}{dt}\frac{e^{-t^2}}{w-t} - \frac{e^{-t^2}}{(w-t)^2}\right] \\ &= \frac{1}{v_T}\left[\frac{e^{-t^2}}{w-t}\bigg|_{-\infty}^{\nu} - \int_{-\infty}^{\nu} dt\frac{e^{-t^2}}{(w-t)^2}\right] = \frac{1}{v_T}\left[\frac{e^{-\nu^2}}{w-\nu} + \frac{d}{dw}\int_{-\infty}^{\nu} dt\frac{e^{-t^2}}{w-t}\right] \\ &= \frac{\exp\left(-\frac{v_{c,\zeta}^2}{v_T^2}\right)}{\omega/k-v_{c,\zeta}} + \frac{1}{v_T}\frac{d}{dw}\Bigg[\underbrace{\int_{-\infty}^{\infty} dt\frac{e^{-t^2}}{w-t}}_{-\sqrt{\pi}Z(w)} - \underbrace{\int_{\nu}^{\infty} dt\frac{e^{-t^2}}{w-t}}_{-\sqrt{\pi}Z(\nu,w)}\Bigg].\end{aligned} \tag{C.18}$$

Plugging in, we find that for a truncated Maxwellian,

$$\frac{4\pi q^2}{k^2 m}\int d^3v \frac{\mathbf{k}\cdot\partial f_s/\partial \mathbf{v}}{\omega - \mathbf{k}\cdot\mathbf{v}} = -\frac{\omega_{ps}^2}{k^2 v_{Ts}^2}\frac{\operatorname{erfc}(-v_{c,\chi}/v_T)\,\operatorname{erfc}(-v_{c,\eta}/v_T)}{4}\big[Z'(w) - Z'(\nu,w)\big]. \tag{C.19}$$

For flowing Maxwellian ions and truncated Maxwellian electrons the plasma dielectric function thus reduces to

$$\hat{\varepsilon}(\mathbf{k},\omega) = 1 - \frac{\omega_{pi}^2}{k^2 v_{Ti}^2}Z'(\xi_i) - \frac{\omega_{pe}^2}{k^2 v_{Te}^2}\frac{\operatorname{erfc}(-v_{c,\chi}/v_T)\,\operatorname{erfc}(-v_{c,\eta}/v_T)}{4}\big[Z'(w) - Z'(\nu,w)\big] \tag{C.20}$$

where $Z(w)$ and $Z(\xi)$ are the conventional plasma dispersion function of arguments w and ξ and $Z(\nu,w)$ is the incomplete plasma dispersion function. Recall that $\xi_i \equiv (\omega - \mathbf{k}\cdot\mathbf{V}_i)/(kv_{Ti})$ and $w \equiv \omega/(kv_{Te})$.

For ion waves where $\xi_i \gg 1$ and $\omega/kv_{Te} \ll 1$, we find

$$\begin{aligned}\omega_\pm = &\left(\mathbf{k}\cdot\mathbf{V}_i \pm \frac{kc_s}{\sqrt{\beta(1-\operatorname{erfc}(\nu)/2) + k^2\lambda_{De}^2}}\right)\left(1 \mp \frac{\sqrt{\pi m_e/8M_i}}{\left(\beta(1-\operatorname{erfc}(\nu)/2) + k^2\lambda_{De}^2\right)^{3/2}}\right) \\ &\mp \frac{\sqrt{\pi}\beta k c_s \delta(w_R - \nu)}{4\left(\beta(1-\operatorname{erfc}(\nu)/2) + k^2\lambda_{De}^2\right)^{3/2}}\end{aligned} \tag{C.21}$$

in which

$$\beta \equiv \frac{\mathrm{erfc}(-v_{c,\chi}/v_{Te})\mathrm{erfc}(-v_{c,\eta}/v_{Te})}{4}. \tag{C.22}$$

In a presheath $\beta \approx 1$, $\nu \approx 3$ and $\delta(w_R - \nu) = 0$, so the last term drops out. The other corrections are very small since $\mathrm{erfc}(\nu)/2 \approx 3 \times 10^{-6}$. Corrections to the conventional ion-acoustic dispersion relation (not accounting for the truncated electron distribution) are thus $\mathcal{O}\left[\exp(-v_{\parallel c}^2/v_{Te}^2)v_{Te}/v_{\parallel c}\right] \ll 1$, which is very small. Thus, the model applied in chapter 4 of flowing Maxwellian ions on stationary Maxwellian electrons accurately describes ion-acoustic instabilities in the presheath.

Appendix D

Two-Stream Dispersion Relation for Cold Flowing Ions

In this appendix, we calculate all four roots of the quartic equation

$$0 = 1 + \frac{1}{k^2\lambda_{De}^2} - \frac{\omega_{p1}^2}{(\omega - \mathbf{k}\cdot\mathbf{V}_1)^2} - \frac{\omega_{p2}^2}{(\omega - \mathbf{k}\cdot\mathbf{V}_2)^2} \tag{D.1}$$

analytically. To do so we use Ferrari's method. Applying the notation that the parallel direction is along the flow, so $\mathbf{k}\cdot\mathbf{V}_1 = k_\parallel V_1$, equation D.1 can be written

$$0 = 1 + k^2\lambda_{De}^2 - \frac{n_1}{n_e}\frac{k^2 c_{s1}^2}{(\omega - k_\parallel V_1)^2} - \frac{n_2}{n_e}\frac{k^2 c_{s2}^2}{(\omega - k_\parallel V_2)^2}. \tag{D.2}$$

Writing this in the standard form

$$A\omega^4 + B\omega^3 + C\omega^2 + D\omega + E = 0 \tag{D.3}$$

yields

$$\begin{aligned}
A &= 1, && \text{(D.4)}\\
B &= -2k_\parallel(V_1 + V_2), && \text{(D.5)}\\
C &= k_\parallel^2(V_2^2 + 4V_1V_2 + V_1^2) - \frac{k^2c_s^2}{1 + k^2\lambda_{De}^2}, && \text{(D.6)}\\
D &= -2k_\parallel^3 V_1V_2(V_1 + V_2) + \frac{2k^2k_\parallel}{1 + k^2\lambda_{De}^2}\left(\frac{n_1}{n_e}c_{s1}^2V_2 + \frac{n_2}{n_e}c_{s2}^2V_1\right), && \text{(D.7)}\\
E &= k_\parallel^4 V_1^2V_2^2 - \frac{k^2k_\parallel^2}{1 + k^2\lambda_{De}^2}\left(\frac{n_1}{n_e}c_{s1}^2V_2^2 + \frac{n_2}{n_e}c_{s2}^2V_1^2\right). && \text{(D.8)}
\end{aligned}$$

Defining u with the substitution $\omega = u - \frac{B}{4A}$, equation D.3 can be written as a depressed quartic

$$u^4 + \alpha u^2 + \beta u + \gamma = 0 \tag{D.9}$$

in which

$$\alpha = -\frac{3B^2}{8A^2} + \frac{C}{A}, \tag{D.10}$$

$$\beta = \frac{B^3}{8A^3} - \frac{BC}{2A^2} + \frac{D}{A} \tag{D.11}$$

$$\gamma = -\frac{3B^4}{256A^4} + \frac{CB^2}{16A^3} - \frac{BD}{4A^2} + \frac{E}{A}. \tag{D.12}$$

For our problem, these coefficients are

$$\alpha = -\frac{1}{2}k_{\parallel}^2\Delta V^2 - \frac{k^2c_s^2}{1+k^2\lambda_{De}^2}, \tag{D.13}$$

$$\beta = \frac{k^2k_{\parallel}}{1+k^2\lambda_{De}^2}\Delta V\left(\frac{n_2}{n_e}c_{s2}^2 - \frac{n_1}{n_e}c_{s1}^2\right), \tag{D.14}$$

$$\gamma = \frac{k_{\parallel}^2\Delta V^2}{4}\left[\frac{1}{4}k_{\parallel}^2\Delta V^2 - \frac{k^2c_s^2}{1+k^2\lambda_{De}^2}\right]. \tag{D.15}$$

Next, the depressed quartic D.9 can be solved using Ferrari's method, which essentially reduces it to solving a cubic equation. To do so, we first add the identity

$$(u^2+\alpha)^2 - u^4 - 2\alpha u^2 = \alpha^2 \tag{D.16}$$

to the depressed quartic, equation D.9, to give

$$(u^2+\alpha)^2 + \beta u + \gamma = \alpha u^2 + \alpha^2. \tag{D.17}$$

This has folded the u^4 term into a perfect square: $(u^2+\alpha)^2$. Next, we want to insert a y into equation D.17 that will fold the right hand side into a perfect square as well. To do this it is convenient to add the identity

$$(u^2+\alpha+y)^2 - (u^2+\alpha)^2 = (\alpha+2y)u^2 - \alpha u^2 + 2y\alpha + y^2 \tag{D.18}$$

to equation D.17 to yield

$$(u^2+\alpha+y)^2 = (\alpha+2y)u^2 - \beta u + (y^2 + 2y\alpha + \alpha^2 - \gamma). \tag{D.19}$$

We have yet to chose y and we want to chose y such that the right hand side of this equation becomes a perfect square. To do this, first note that if you expand a perfect square

$$(su+t)^2 = (s^2)u^2 + (2st)u + (t^2) \tag{D.20}$$

that the square of the second coefficient minus 4 times the product of the first and third coefficients vanishes

$$(2st)^2 - 4(s^2)(t^2) = 0. \tag{D.21}$$

So for equation D.19, we should define y to solve

$$(-\beta)^2 - 4(2y+\alpha)(y^2 + 2y\alpha + \alpha^2 - \gamma) = 0 \tag{D.22}$$

which can be written

$$y^3 + \frac{5}{2}\alpha y^2 + (2\alpha^2 - \gamma)y + \left(\frac{\alpha^3}{2} - \frac{\alpha\gamma}{2} - \frac{\beta^2}{8}\right). \tag{D.23}$$

With the definition for y given by equation D.22, equation D.19 can be written

$$(u^2 + \alpha + y)^2 = \left(u\sqrt{\alpha + 2y} - \frac{\beta}{2\sqrt{\alpha + 2y}}\right)^2. \tag{D.24}$$

Taking the square root of both sides and rearranging gives

$$u^2 + \left(\mp_s \sqrt{\alpha + 2y}\right)u + \left(\alpha + y \pm_s \frac{\beta}{2\sqrt{\alpha + 2y}}\right) = 0. \tag{D.25}$$

Which can be easily solved with the quadratic equation to give

$$u = \pm_s \frac{1}{2}\sqrt{\alpha + 2y} \pm_t \frac{1}{2}\sqrt{\alpha + 2y - 4\left(\alpha + y \pm_s \frac{\beta}{2\sqrt{\alpha + 2y}}\right)} \tag{D.26}$$

in which the s and t subscripts denote the dependent and independent $\pm$'s.

We just need one of the three values of y from the cubic equation D.23, it does not matter which, and we have our fourth order equation solved. There is a similar, but more brief, method for solving the cubic equation, but I'll just quote the results. The solutions of

$$y^3 + ay^2 + by + c = 0 \tag{D.27}$$

are

$$y = -\frac{P}{3U} + U - \frac{a}{3} \tag{D.28}$$

in which

$$P = b - \frac{a^2}{3}, \tag{D.29}$$

$$Q = c + \frac{2a^3 - 9ab}{27}, \tag{D.30}$$

$$U = \left(-\frac{Q}{2} \pm \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}\right)^{1/3} \tag{D.31}$$

in which the sign before the square root does not matter since both give you the same answer. The 1/3 power leads to the three solutions.

For our problem

$$a = \frac{5}{2}\alpha, \tag{D.32}$$

$$b = 2\alpha^2 - \gamma, \tag{D.33}$$

$$c = \frac{\alpha^3}{2} - \frac{\alpha\gamma}{2} - \frac{\beta^2}{8}, \tag{D.34}$$

so

$$P = -\frac{\alpha^2}{12} - \gamma, \tag{D.35}$$

$$Q = -\frac{\alpha^3}{108} + \frac{\alpha\gamma}{3} - \frac{\beta^2}{8}. \tag{D.36}$$

Plugging in for our values of α, β, and γ, we find

$$a = -\frac{5}{4}k_\parallel^2\Delta V^2 - \frac{5}{2}\frac{k^2c_s^2}{1+k^2\lambda_{De}^2}, \tag{D.37}$$

$$b = \frac{7}{16}k_\parallel^4\Delta V^4 + \frac{9}{4}k_\parallel^2\Delta V^2\frac{k^2c_s^2}{1+k^2\lambda_{De}^2} + 2\frac{k^4c_s^4}{(1+k^2\lambda_{De}^2)^2}, \tag{D.38}$$

$$c = -\frac{3}{64}k_\parallel^6\Delta V^6 - \frac{13}{32}k_\parallel^4\Delta V^4\frac{k^2c_s^2}{1+k^2\lambda_{De}^2} \tag{D.39}$$

$$- \; k_\parallel^2\Delta V^2\frac{k^4}{(1+k^2\lambda_{De}^2)^2}\left(c_s^2 - \frac{1}{2}\frac{n_1n_2}{n_e^2}c_{s1}^2c_{s2}^2\right) - \frac{1}{2}\frac{k^6c_s^6}{(1+k^2\lambda_{De}^2)^3}, \tag{D.40}$$

$$P = -\frac{1}{12}\left(k_\parallel^2\Delta V^2 - \frac{k^2c_s^2}{1+k^2\lambda_{De}^2}\right)^2, \tag{D.41}$$

$$Q = -\frac{1}{108}\left[k_\parallel^2\Delta V^2 - \frac{k^2c_s^2}{1+k^2\lambda_{De}^2}\right]^3 + \frac{1}{2}k_\parallel^2\Delta V^2\frac{n_1}{n_e}\frac{n_2}{n_e}\frac{k^4c_{s1}^2c_{s2}^2}{(1+k^2\lambda_{De}^2)^2}. \tag{D.42}$$

Using these values gives

$$\frac{Q^2}{4} + \frac{P^3}{27} = -\frac{1}{432}k_\parallel^2\Delta V^2\frac{n_1n_2}{n_e^2}\frac{k^4c_{s1}^2c_{s2}^2}{(1+k^2\lambda_{De}^2)^2} \times \left[\left(k_\parallel^2\Delta V^2 - \frac{k^2c_s^2}{1+k^2\lambda_{De}^2}\right)^3 \right. \tag{D.43}$$

$$\left. - \; 27k_\parallel^2\Delta V^2\frac{n_1n_2}{n_e^2}\frac{k^4c_{s1}^2c_{s2}^2}{(1+k^2\lambda_{De}^2)^2}\right]$$

and

$$U = \left(\frac{1}{216}\left[k_\parallel^2\Delta V^2 - \frac{k^2c_s^2}{1+k^2\lambda_{De}^2}\right]^3 - \frac{1}{4}k_\parallel^2\Delta V^2\frac{n_1}{n_e}\frac{n_2}{n_e}\frac{k^4c_{s1}^2c_{s2}^2}{(1+k^2\lambda_{De}^2)^2}\right. \tag{D.44}$$

$$\left. \pm \; \frac{\sqrt{3}}{36}k_\parallel\Delta V\frac{\sqrt{n_1n_2}}{n_e}\frac{k^2c_{s1}c_{s2}}{1+k^2\lambda_{De}^2}\sqrt{27k_\parallel^2\Delta V^2\frac{n_1n_2}{n_e^2}\frac{k^4c_{s1}^2c_{s2}^2}{(1+k^2\lambda_{De}^2)^2} - \left(k_\parallel^2\Delta V^2 - \frac{k^2c_s^2}{1+k^2\lambda_{De}^2}\right)^3}\right)^{1/3}$$

We now have all the pieces, and need to plug into

$$\omega = -\frac{B}{4A} \pm_s \frac{1}{2}\sqrt{\alpha + 2y} \pm_t \frac{1}{2}\sqrt{-3\alpha - 2y \mp_s \frac{2\beta}{\sqrt{\alpha + 2y}}}. \tag{D.45}$$

However, U and thus y are very complicated and the exact answer for ω is so arduous that is essentially unusable. So, we will consider two limiting cases

$$k_\parallel \Delta V \ll \frac{kc_s}{\sqrt{1 + k^2\lambda_{De}^2}} \quad \text{and} \quad k_\parallel \Delta V \gg \frac{kc_s}{\sqrt{1 + k^2\lambda_{De}^2}}. \tag{D.46}$$

D.1 Small Flow Difference

Here, we expand in the limit

$$k_\parallel \Delta V \ll \frac{kc_s}{\sqrt{1 + k^2\lambda_{De}^2}}. \tag{D.47}$$

First, keep up to order $\mathcal{O}(\epsilon)$ where $\epsilon = k_\parallel(V_1 - V_2)$. Then,

$$\alpha = -\frac{k^2 c_s^2}{1 + k^2\lambda_{De}^2} + \mathcal{O}(\epsilon^2), \tag{D.48}$$

$$\beta = k_\parallel \Delta V \frac{k^2\left(\frac{n_2}{n_e}c_{s2}^2 - \frac{n_1}{n_e}c_{s1}^2\right)}{1 + k^2\lambda_{De}^2}, \tag{D.49}$$

$$P = -\frac{1}{12}\frac{k^4 c_s^4}{(1 + k^2\lambda_{De}^2)^2} + \mathcal{O}(\epsilon^2), \tag{D.50}$$

$$a = -\frac{5}{2}\frac{k^2 c_s^2}{1 + k^2\lambda_{De}^2} + \mathcal{O}(\epsilon^2), \tag{D.51}$$

$$U = \frac{kc_s}{\sqrt{1 + k^2\lambda_{De}^2}}\left[-\frac{1}{216}\frac{k^3 c_s^3}{(1 + k^2\lambda_{De}^2)^{3/2}} \pm \frac{\sqrt{3}}{36}k_\parallel \Delta V \frac{\sqrt{n_1 n_2}}{n_e}\frac{k^2 c_{s1}c_{s2}}{1 + k^2\lambda_{De}^2}\right]^{1/3} + \mathcal{O}(\epsilon^2). \tag{D.52}$$

Using the series expansion

$$(a \pm \epsilon)^{1/3} = a^{1/3} \pm \frac{1}{3}\frac{1}{a^{2/3}}\epsilon + \mathcal{O}(\epsilon^2) \tag{D.53}$$

and $(-1)^{1/3} = (1/2 + i\sqrt{3}/2)$, gives

$$U = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\frac{kc_s}{\sqrt{1 + k^2\lambda_{De}^2}}\left(\frac{1}{6}\frac{kc_s}{\sqrt{1 + k^2\lambda_{De}^2}} \mp \frac{\sqrt{3}}{3}k_\parallel \Delta V \frac{\sqrt{n_1 n_2}}{n_e}\frac{c_{s1}c_{s2}}{c_s^2}\right) + \mathcal{O}(\epsilon^2) \tag{D.54}$$

The above give

$$y = \frac{k^2 c_s^2}{1 + k^2\lambda_{De}^2} \mp i\frac{kc_s}{\sqrt{1 + k^2\lambda_{De}^2}}k_\parallel \Delta V \frac{\sqrt{n_1 n_2}}{n_e}\frac{c_{s1}c_{s2}}{c_s^2}. \tag{D.55}$$

In this limit, we get two always stable roots

$$\omega_{\substack{1\\2}} = \frac{1}{2}k_\parallel(V_1+V_2) \pm \frac{kc_s}{\sqrt{1+k^2\lambda_{De}^2}} - \frac{1}{2}k_\parallel \Delta V\left(\frac{n_2}{n_e}\frac{c_{s2}^2}{c_s^2} - \frac{n_1}{n_e}\frac{c_{s1}^2}{c_{s2}^2}\right) \tag{D.56}$$

and two other roots, only one of which can be unstable

$$\omega_{\substack{3\\4}} = \frac{1}{2}k_\parallel(V_1+V_2) + \frac{1}{2}k_\parallel \Delta V\left(\frac{n_2}{n_e}\frac{c_{s2}^2}{c_s^2} - \frac{n_1}{n_e}\frac{c_{s1}^2}{c_{s2}^2}\right) \pm ik_\parallel \Delta V\frac{\sqrt{n_1 n_2}}{n_e}\frac{c_{s1}c_{s2}}{c_s^2}. \tag{D.57}$$

D.2 Large Flow Difference

Next, we'll consider the other limit

$$k_\parallel \Delta V \gg \frac{kc_s}{\sqrt{1+k^2\lambda_{De}^2}}. \tag{D.58}$$

First, we keep up to order $\mathcal{O}(\epsilon^2)$ where $\epsilon = kc_s/\sqrt{1+k^2\lambda_{De}^2}$. Then,

$$\begin{aligned}
\alpha &= -\frac{1}{2}k_\parallel^2\Delta V^2 - \frac{k^2c_s^2}{1+k^2\lambda_{De}^2}, && \text{(D.59)}\\
\beta &= \frac{k_\parallel \Delta V}{1+k^2\lambda_{De}^2}k^2\left(\frac{n_2}{n_e}c_{s2}^2 - \frac{n_1}{n_e}c_{s1}^2\right), && \text{(D.60)}\\
P &= -\frac{1}{12}\left(k_\parallel^4\Delta V^4 - 2k_\parallel^2\Delta V^2\frac{k^2c_s^2}{1+k^2\lambda_{De}^2} + \mathcal{O}(\epsilon^4)\right), && \text{(D.61)}\\
a &= -\frac{5}{4}k_\parallel^2\Delta V^2 - \frac{5}{2}\frac{k^2c_s^2}{1+k^2\lambda_{De}^2}, && \text{(D.62)}\\
U &= \frac{1}{6}k_\parallel^2\Delta V^2 - \frac{1}{216}\frac{k^2c_s^2}{1+k^2\lambda_{De}^2} \pm i\frac{\sqrt{3}}{108}\frac{\sqrt{n_1n_2}}{n_e}\frac{k^2c_{s1}c_{s2}}{1+k^2\lambda_{De}^2}. && \text{(D.63)}
\end{aligned}$$

Plugging in the above gives

$$y = \frac{3}{4}k_\parallel^2\Delta V^2 + \frac{1}{2}\frac{k^2c_s^2}{1+k^2\lambda_{De}^2}. \tag{D.64}$$

We find that all four roots are real in this case

$$\omega_{\substack{1\\2}} = \frac{1}{2}k_\parallel V_1 \pm \frac{\sqrt{2}}{4}\frac{kc_s}{\sqrt{1+k^2\lambda_{De}^2}} \pm \frac{\sqrt{2}}{4}\frac{k^2\lambda_{De}^2}{\sqrt{1+k^2\lambda_{De}^2}}\frac{\omega_{p1}^2}{kc_s} \tag{D.65}$$

and

$$\omega_{\substack{3\\4}} = \frac{1}{2}k_\parallel V_2 \pm \frac{\sqrt{2}}{4}\frac{kc_s}{\sqrt{1+k^2\lambda_{De}^2}} \pm \frac{\sqrt{2}}{4}\frac{k^2\lambda_{De}^2}{\sqrt{1+k^2\lambda_{De}^2}}\frac{\omega_{p2}^2}{kc_s}. \tag{D.66}$$

These look like modified acoustic waves emitting from each of the two beams.

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